STABILITY OF HETEROGENEOUS SWIRLING FLOWS

By

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To my Wife and Parents

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LIST OF SYMBOLS

(Arranged in alphabetical order)

c	complex axial wave velocity, $c = \frac{\omega}{k}$
$\hat{c}$	complex azimuthal wave velocity, $\hat{c} = \frac{\omega}{m}$
D	$\frac{d}{dr}$
D^{*}	$D + \frac{1}{r}$
D_{*}	$D - \frac{1}{r}$
d_{*}	$\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \theta} + W\frac{\partial}{\partial z}$
E	$(1 + s^2)^{-1}$
g	gravitational constant
$I_m(z)$	modified Bessel function of the first kind
$K_m(z)$	modified Bessel function of the second kind
k	axial wave number
m	azimuthal wave number
N	$kW + m\Omega - \omega$
n_j	N_j/Ω_j
P	total pressure
P_o	steady-state pressure
p	amplitude of $\hat{p}$

LIST OF SYMBOLS (Continued)

$\hat{p}$	perturbation pressure
q	$\sqrt{1 - (4 + \sigma)/n_j^2}$ (in Section 5.2 and 5.3)
q_j	$\sqrt{1 - 4/n_j^2}$ (in Section 5.1)
$\hat{q}$	$\sqrt{1 - 4/n_o^2}$ (in Section 5.3)
R_j	radial position
r	radial co-ordinate in cylindrical co-ordinate system
s	asymmetry parameter, $s = m/kr$
T	surface tension
t	time
u	amplitude of $\hat{u}$
$\hat{u}$	radial component of perturbation velocity
$\bar{u}$	amplitude of $\hat{u}$ (for axisymmetric disturbances only)
v	amplitude of $\hat{v}$
$\hat{v}$	tangential component of perturbation velocity
W	axial velocity component (steady-state)
w	amplitude of $\hat{w}$
$\hat{w}$	axial component of perturbation velocity
$\bar{w}$	amplitude of $\hat{w}$ (for axisymmetric disturbances only)
z	axial co-ordinate of cylindrical co-ordinate system

LIST OF SYMBOLS (Continued)

Greek letters

α	density ratio at interface, $\alpha = \rho_2/\rho_1$
β	angular velocity ratio at interface, $\beta = \Omega_2/\Omega_1$
Γ	$T/\rho_1 k^2 R^3$
Γ_j	$T/\rho_j \Omega_j^2 R^3$
γ_j	$k(W_2 - W_1)/\Omega_j$
δ	width of middle ring
ε	δ/R
ε_1	inner boundary ratio, $\varepsilon_1 = R_1/R$
ε_2	outer boundary ratio, $\varepsilon_2 = R/R_2$
ζ	index
η	amplitude of $\hat{\eta}$
$\hat{\eta}$	radial perturbation displacement
θ	azimuthal co-ordinate of cylindrical co-ordinate system
κ	kR
Λ	$2(m/n_o) - (m/n_o)^2$
λ	$(1 + s^2)\ \Phi/\rho_o[DW + s(rD\Omega + 4\Omega)]^2$
μ	$\lvert 1 - \sigma \rvert$
ν	$\sqrt{m^2 + \sigma\Lambda + (\sigma/2)^2}$

ρ amplitude of $\hat{\rho}$

$\hat{\rho}$ perturbation density

ρ_o steady-state density

σ density parameter $[\rho_o(r) \sim r^{\sigma}]$

τ axial velocity ratio at interface, $\tau = W_2/W_1$

Φ $D[\rho_o(r^2\Omega)^2]/r^3$

ϕ u r

χ $q_j kr$

Ψ Stoke's stream function $\Psi = \Psi(r, z)$

$\hat{\Psi}$ Stoke's stream function $\hat{\Psi} = \hat{\Psi}(r, z; t)$

ψ $N^{-\zeta}u$

Ω angular velocity component (steady-state)

ω complex amplification factor, $\omega = \omega_r + i\omega_i$

Subscripts

c critical

i imaginary part

j index

r real part

Symbols not listed here have all been explained in the text at the time of their first usage.

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STABILITY OF HETEROGENEOUS SWIRLING FLOWS

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A general mathematical investigation of the hydrodynamic stability of heterogeneous swirling flows with possible applications to the gaseous centrifuge method for separation of uranium isotopes and to the proposed vortex-driven gas-core nuclear reactor is presented. The flow under consideration is assumed to be inviscid and incompressible but has radially dependent density and velocity components within an annular region between two concentric cylinders. Gravitational effects are neglected. The concept of "centrifugal stability" is introduced to distinguish instabilities caused by centrifugal forces due to rotation of the flow from those caused by shearing effects due to the relative motion between two adjacent fluid layers. A flow distribution is "centrifugally stable" if the product of its density and the square of its circulation is an

increasing function of radius. It should be noted that the classical Rayleigh-Synge criterion applies only to centrifugally induced instabilities. A generalized sufficiency condition for hydrodynamic stability of swirling flows against arbitrary infinitesimal perturbations is obtained. The sufficiency condition states that a non-dimensional parameter λ, reminiscent of the familiar Richardson number encountered in two-dimensional stratified shear flows, shall be greater or equal to a quarter at all points within the flow for stability.

Extensions of Howard's semi-circle theorem are obtained for four types of swirling flows with centrifugally stable velocity and density distributions. The results are compared with the exact solutions obtained for three special velocity and density profiles. A semi-circle bound for instability, growth rates and phase velocities is also obtained for solid body rotation with arbitrary density distribution when the flow is subject to azimuthally periodic disturbances.

Two interfacial conditions for discontinuous density and velocity distributions are obtained via both mathematical and physical arguments. In particular, the centrifugal effects due to discontinuities cannot be neglected when matching techniques are applied at interfaces. Effects of surface tension on stability characteristics, which are

found to always stabilize the flows except for axisymmetric modes with axial wave numbers less than a certain value, become important for flows with one or more interfaces.

Four types of velocity and density profiles are examined analytically, and exact solutions expressed in terms of the modified Bessel functions are obtained. Stability characteristics of these flows are found and shown to always satisfy the general criteria derived earlier. It is found that pure rotational flows without axial velocity gradients with large shear are more unstable to two-dimensional azimuthally periodic disturbances than either axisymmetric or more general three-dimensional disturbances. Furthermore, solutions with large wave numbers are most unstable for velocity profiles containing one or more vortex sheets.

Ulrich H Kurzweg
Chairman

CHAPTER 1

INTRODUCTION

1.1 Physical Considerations

A recent report by Abajian and Fishman (1973) on the supply of enriched uranium indicates the possible importance of the high-speed gaseous centrifuge process for separation of uranium isotopes. Isotope U^{235} is separated from isotope U^{238} in a high-speed rotating cylinder because of the slight difference in molecular weights. The gas molecules experience outward centrifugal forces produced by the rotation, and are subject to thermal diffusive forces which tend to keep the molecules evenly distributed. Since the diffusion effect is smaller for the heavier molecules there are relatively fewer specimens of the heavier isotope at the inner radius. A major, and as yet incompletely considered problem, associated with the centrifugal separation process concerns the hydrodynamic stability of this flow. For isotope separation it is clearly a requirement that the flow be stable for all possible hydrodynamic disturbances.

The need for a stability analysis on a rotating flow of variable density is also seen in the vortex-driven

gas-core nuclear reactor proposed by the Oak Ridge National Laboratory. In this method for generating nuclear power, a gas of low molecular weight is injected tangentially into a cylindrical chamber along with fissionable materials also in gaseous form. The gas is withdrawn either through a central porous tube, or through the open ends of a central cylinder, thus generating a potential vortex inside the chamber. The gaseous fuel being of heavier molecular weight moves radially outward, while the light gas swirls radially inward and moves toward the exit. An annular cloud of gaseous fuel is then accumulated at some radial position, and will become heat producing when the neutron flux of the cloud achieves a certain critical value. Again this is a flow situation which requires careful consideration for the condition of stability.

The recent speculations by Soviet and American researchers on the possibility of achieving a fusion reaction with high-powered lasers has led to several proposals for controlling such reactions. One such design for a laser fusion reactor involves containment of a laser imploded hydrogen pellet with a liquid lithium vortex, the liquid lithium being fed tangentially into a spherical pressure vessel. A free-standing vortex is formed in the center as a result of the high velocity spiralling lithium. During the fusion process, the energy released by the fused

deuterium is absorbed in this liquid lithium vortex. Due to the properties of metal, the lithium near the center of the vortex where the fusion takes place has a higher temperature and a lower density. The hot lithium is thus kept in the center of the spherical pressure vessel by essentially centrifugal effects. As a result the heat produced can be transferred by extracting the high-temperature liquid lithium through the bottom of the vessel. This example of a heterogeneous swirling flow is different physically from the previous two, but they all show very similar hydrodynamic conditions which need to be examined for their hydrodynamic stability characteristics. We would like to carry out such a stability analysis for heterogeneous swirling flows.

1.2 Mathematical Model

The study of the hydrodynamic stability for heterogeneous incompressible swirling flows with radially dependent angular and axial velocity components and density $\rho_0(r)$ is of considerable practical interest in connection with the just-mentioned physical problems. The swirling stability of such flows is characterized by the exchange of position between two fluid elements in a centrifugal force field, i.e., the transfer of energy from the mean velocity field to small amplitude perturbations in the

form of travelling waves. The instability mechanism is similar to that characterizing the stability of heterogeneous two-dimensional parallel shear flows. In such shear flows the stability is governed by the magnitude of the ratio of the density gradient to the shearing rate, the Richardson number. To understand the behavior of this instability mechanism in swirling flows, it is necessary to consider the mathematics of the Navier-Stokes equations, the continuity equation, and the condition of incompressibility and solve these governing differential equations.

The onset of instability can be determined by assuming temporal or spatial disturbances whose behavior is governed by the linearized forms of the governing non-linear equations. However, one should keep in mind that instability does not necessarily transfer a laminar flow into turbulence. Unstable modes sometimes are simply a transition from one type of laminar motion to another. Discussion of the stability beyond the initial onset is not within the capability of a linear analysis and will not be considered here.

It has been accepted that the method of normal modes is complete and well explains the linear wave behavior in fluid mechanics (see Lin 1955, Roberts 1967, Betchov and Criminale 1967). The linearized partial differential equations under consideration are cyclic with respect to

time and to two spatial variables. This strongly suggests they be solved by Laplace (time) and Fourier (space) transform methods (Case 1960 a, b). The differential equations after transformation are ordinary and in general complex both with respect to the temporal and spatial parameters (Gaster 1963). If any one of the Laplace or Fourier modes is unstable, we consider the system to be unstable as a whole. In other words, this method is assuming that each perturbation can be solved into dynamically independent wave components. In our analysis we will restrict ourselves only to temporal perturbations.

In the case of an inviscid analysis as will be carried out here, the eigenvalue problem will become singular, and the singularity admits solutions with discontinuous derivatives and a continuous spectrum of eigenvalues in addition to well-behaved solutions with a discrete spectrum (Lamb 1932, §151). This singularity may be regarded as a consequence of either our assumption of periodic wave motions or our neglect of viscous (or diffusion) effects. Two difficulties arise as a result of such an analysis: first, a critical layer is introduced inside the flow domain, which can be smoothed out only by addition of viscosity (or diffusion); second, the solution to the governing equations has as a complex eigen wave speed, $c = c_r + ic_i$, and its complex conjugate. That is, both damped and growing

waves exist simultaneously for the same wave speed and it accordingly becomes much more difficult to locate the stability boundary (refer to Chapter 2 for definition) except for some special profiles in which the principle of exchange stabilities is valid (Howard 1963).

1.3 Previous Work

It was early shown by Rayleigh (1916) that the inviscid stability of fluid motion in cylindrical strata under axisymmetric disturbances required only that the square of the circulation increase outwards. His argument was based on an analogy between the motion and the equilibrium of a heterogeneous liquid under gravity and was accomplished by the consideration of the increment in kinetic energy consequent on an interchange of two rings of liquid. Under this criterion, a homogeneous potential vortex without axial flow gradient is neutrally stable against axisymmetric disturbances. Von Kármán (1934) gave a slightly different physical derivation of the Rayleigh criterion by letting the centrifugal force balance the pressure gradient established by the velocity field.

Synge (1933) gave a rigorous proof of the Rayleigh criterion by taking into account the radially dependent density variation and casting the problem into the form of a Sturm-Liouville system. The well-known Rayleigh-Synge

criterion then states that an inviscid rotating flow is stable against axisymmetric disturbances whenever the product of its density and the square of its circulation is an increasing function of radius.

Ponstein (1959) first looked at the stability of rotating cylindrical jets for non-rotationally symmetric perturbations. His calculations showed that in some cases non-axisymmetric modes were even more unstable than the axisymmetric ones, hence it was important to take them into consideration. He also found that the surface tension in general stabilized the fluid motion, except for axisymmetric perturbations with axial wave numbers less than a certain value. This result was first shown by Rayleigh (1879) (also Lamb 1932, §273).

Chandrasekhar (1960a, b; 1961, §67, §78b) tried to generalize the Rayleigh-Synge criterion for arbitrary disturbances and for rotating flows in the presence of an axial flow gradient. He came to the conclusion that the Rayleigh-Synge criterion for stability continued to be valid irrespective of perturbation and axial flow conditions. This would mean that a behavior resembling the Squire's (1933) transformation for plane parallel flows with respect to three-dimensional perturbations could be expected in the swirling flow case. However, his results appear rather implausible because of the characteristic

difference between the axisymmetric perturbations for rotating flows and the non-axisymmetric ones for swirling flows. In a mathematical sense, the former is governed by a Sturm-Liouville system where the principle of exchange stabilities is usually expected, while the latter in general is not. In a physical sense, the former concerns an instability of the Rayleigh-Taylor type, in which particle exchange is restricted in axial direction ($m = 0$), while the latter is an instability of the Kelvin-Helmholtz together with Rayleigh-Taylor type, in which particle exchanges occur generally along the direction of the mean swirling flows. Howard (1962) first pointed out that Chandresakhar's extension of Rayleigh's criterion was questionable and later Fung and Kurzweg (1973) gave an example which clearly showed that Chandrasekhar's result was incorrect.

Alterman (1961) studied the instabilities of cylindrical liquid jets bounded by a surrounding liquid and subject to rotation. He found that the rotation-induced instability for axisymmetric disturbances depended on the angular velocities of the jet relative to its surroundings.

Batchelor and Gill (1962) made a general mathematical analysis of the stability of axisymmetric jets for homogeneous unbounded fluids. A condition similar to Rayleigh's (1880) inflection point theory was shown to be

necessary for instability and also a semi-circle theorem was found to offer a bound for possible unstable waves.

Following an important paper by Miles (1961) and a note by Howard (1961) on the stability of heterogeneous shear flows, Howard and Gupta (1962) obtained two stability conditions for axisymmetric disturbances; one for pure axial flow with radially dependent density and another for swirling flows with constant density. In both cases they found that a sufficient condition for stability required that a non-dimensional quantity, reminiscent of the Richardson number, be greater than 1/4 anywhere within the flow region, and furthermore, that a semi-circle theorem held for "statically stable" profiles where the Rayleigh-Synge criterion was satisfied. They also considered flow stability against non-axisymmetric disturbances, but no general conclusion could be drawn from their stability equation in this case as the sufficiency condition was found to be violated for sufficiently small axial wave numbers. This result does, however, suggest that azimuthally periodic perturbations are more unstable.

In a paper concerning two-dimensional vortex-type flows with constant density, Michalke and Timme (1967) derived a stability equation in terms of perturbation vorticity. Two analytical and one numerical solutions were found for three types of vortex flows. Their results revealed

that the two-dimensional azimuthally periodic disturbances were apparently amplified more strongly than the corresponding three-dimensional ones. However, in their first example of the cylindrical vortex sheet, they failed to consider the centrifugal effects due to the discontinuity of velocity at the interface, and accordingly their result must be modified. An analytical example by Fung and Kurzweg (1973) also showed that the flow was less stable with respect to azimuthally periodic disturbances than to the axisymmetric ones.

Weske and Rankin (1963) investigated experimentally the motion in the region of the core of a vortex which exhibited peripheral vorticity. The early stages of instability observed are essentially two-dimensional with large azimuthal wave numbers.

Recently, Johnston(1972) observed one type of instability in rotating flows by injecting a light gas tangentially into a stationary gas of higher density. The onset of instability is characterized by the two-dimensional azimuthally periodic modes which are apparently most unstable (see Kurzweg 1974).

Leibovich (1969) generalized Howard and Gupta's (1962) stability condition to swirling flows with radially dependent density but restricted his discussions to axisymmetric perturbations.

Kurzweg (1969) derived a general stability equation for heterogeneous swirling flows with radially dependent velocity and density for arbitrary infinitesimal disturbances in the absence of viscous and gravitational effects. A sufficient condition for stability was found and in particular it was shown that such heterogeneous swirling flows could be guaranteed stable against arbitrary infinitesimal disturbances when the density was an increasing function of radius and at the same time the velocity gradients remained small. Later Kurzweg (1970) obtained the bounds on growth rate and phase velocity for the inviscid ability of a constant density swirling flow with the narrow gap approximation. Two types of swirling profile were investigated numerically and the effect of adding an axial velocity to the Couette flow was also discussed.

Here, we would like to generally investigate the stability characteristics of heterogeneous swirling flows and compare our results with the existing theoretical and experimental ones.

CHAPTER 2

MATHEMATICAL FORMULATION

We wish to examine the stability of swirling flows between two concentric cylinders $[R_1, R_2]$ having the steady-state radially dependent velocity field $\vec{V}\ [0, r\Omega(r), W(r)]$. The fluid under consideration is assumed inviscid, incompressible and non-heat-conducting but has a radially dependent density $\rho_0(r)$. Euler's equations of motion, the incompressibility condition and the continuity equation governing such a swirling flow in cylindrical co-ordinates (r, θ, z) are, respectively,

$$\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}\right) = -\frac{1}{\rho}\nabla P + \vec{g} \qquad (2.1)$$

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot (\nabla \rho) = 0 \qquad (2.2)$$

$$\nabla \cdot \vec{V} = 0 \qquad (2.3)$$

To examine the stability of this heterogeneous flow, we postulate an initial set of solutions based on the summation of the steady-state flow and infinitesimal time-dependent fluctuations, namely,

$$
\begin{aligned}
V_r &= \hat{u}(r,\theta,z;t) \\
V_\theta &= r\Omega(r) + \hat{v}(r,\theta,z;t) \\
V_z &= W(r) + \hat{w}(r,\theta,z;t) \\
P &= P_o(r) + \hat{p}(r,\theta,z;t) \\
\rho &= \rho_o(r) + \hat{\rho}(r,\theta,z;t)
\end{aligned}
\tag{2.4}
$$

Substituting equations (2.4) into equations (2.1), (2.2) and (2.3), neglecting the gravitational effects, and following the normal linearization procedure of dropping second and higher order terms in the assumed infinitesimal perturbations lead to the linearized differential equations

$$\frac{\partial \hat{p}}{\partial r} + \rho_o(d_* \hat{u} - 2\Omega \hat{v}) - \hat{\rho} r \Omega^2 = 0 \tag{2.5a}$$

$$\frac{\partial \hat{p}}{\partial \theta} + \rho_o r[d_* \hat{v} + D^*(r\Omega)\hat{u}] = 0 \tag{2.5b}$$

$$\frac{\partial \hat{p}}{\partial z} + \rho_o[d_* \hat{w} + (DW)\hat{u}] = 0 \tag{2.5c}$$

$$d_* \hat{\rho} + (D\rho_o)\hat{u} = 0 \tag{2.5d}$$

$$\frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{1}{r}\frac{\partial \hat{v}}{\partial \theta} + \frac{\partial \hat{w}}{\partial z} = 0 \tag{2.5e}$$

where

$$D = \frac{d}{dr}\ , \qquad D^{*} = D + \frac{1}{r}$$

and

$$d_{*} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta} + W\frac{\partial}{\partial z}$$

with boundary conditions $\hat{u} = 0$ at R_1 and R_2.

Eliminating $\hat{v}$, $\hat{w}$ and $\hat{p}$ from equations (2.5), we obtain the following two equivalent partial differential equations for stability.

$$\frac{1}{r^2}\frac{\partial^2\hat{p}}{\partial\theta^2} + \frac{\partial^2\hat{p}}{\partial z^2} - \rho_o\left[d_{*}(D^{*}\hat{u}) - \frac{D^{*}(r\Omega)}{r}\frac{\partial\hat{u}}{\partial\theta} - (DW)\frac{\partial\hat{u}}{\partial z}\right] = 0 \qquad (2.6)$$

$$d_{*}\left(\frac{\partial\hat{p}}{\partial r}\right) + \frac{2\Omega}{r}\frac{\partial\hat{p}}{\partial\theta} + \rho_o d_{*}^{2}\hat{u} + \Phi\hat{u} = 0 \qquad (2.7)$$

where

$$\Phi = \frac{D[\rho_o(r^2\Omega)^2]}{r^3}$$

The coefficients of equations (2.5) or equations (2.6) and (2.7) depend only on the radial coordinate r, while the remaining independent variables z, θ and t enter only in the partial derivatives. This allows one to assume that (2.5), (2.6) and (2.7) admit solutions having an exponential behavior in z, θ , and t. This assumption reduces all these partial differential equations to ordinary differential equations.

Following the normal mode analysis indicated in Chapter 1, we apply the Laplace transform with respect to time t and the Fourier transform with respect to the spatial variables (z, θ). The transforms have the form

$$\{p,u\}(k,m,\omega;r) = \int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{+\infty} \{\hat{p},\hat{u}\}(r,z,\theta;t)e^{i(kz+m\theta-\omega t)}dz\,d\theta\,dt \tag{2.8}$$

which is equivalent to the standard transformation of assuming

$$\{\hat{p},\hat{u}\} = \{p(r),u(r)\}e^{i(kz+m\theta-\omega t)}$$

We are here considering only temporal disturbances so that the axial wave number k is real while the amplification

factor $\omega = \omega_r + i\omega_i$ is complex; m represents the azimuthal wave number and takes on only integer values because of the periodicity of the solution in the θ direction. Using transform (2.8) in equations (2.6) and (2.7) leads to the two ordinary differential equations

$$(k^2 + \frac{m^2}{r^2})p = i\rho_0\{[k(DW) + \frac{mD^*(r\Omega)}{r}]u - N(D^*u)\} \tag{2.9}$$

$$NDp + \frac{2m\Omega}{r}p = i(\Phi - \rho_0 N^2)u \tag{2.10}$$

By eliminating p, these equations may be reduced, after some manipulation, to the single second order stability equation

$$N^2[D(\rho_0 ED^*) - \rho_0 k^2]u + u[NF + G] = 0 \tag{2.11}$$

with boundary conditions $u(R_1) = u(R_2) = 0$

Here

$$N = kW + m\Omega - \omega$$

$$s = \frac{m}{kr}$$

$$E = (1 + s^2)^{-1}$$

$$F = -kD_*\{\rho_0 E[DW + sD^*(r\Omega)]\}$$

$$G = k^2 E[\Phi - 2s\rho_0\Omega(DW) + s^2 r\Omega^2(D\rho_0)]$$

$$D_* = D - \frac{1}{r}$$

Equation (2.11) was first derived by Kurzweg (1969) and reduces to the one given by Howard and Gupta in the limiting case of constant density. Together with the homogeneous boundary conditions, equation (2.11) forms an eigenvalue problem for the complex eigenvalue ω for given specific values of m, k, Ω, W and ρ_0. The system is unstable whenever $\omega_i > 0$. However, one should keep in mind that since (2.11) is invariant under complex conjugation, any solution with $\omega_i < 0$ (damped wave) implies the existence of another solution with $\omega_i > 0$ (growing wave), and hence instability. This fact shows that in the absence of viscous effects there can be no damped wave and the flow is either unstable ($\omega_i > 0$) or neutrally stable ($\omega_i = 0$). In order to distinguish the bounding neutral curve ($\omega_i = 0$) characterizing stability from the other neutral curves (also $\omega_i = 0$) for all other stable solutions, we define, following Miles (1961), the neutral curve as a stability boundary if and only if there exist contiguous eigenvalues for which $\omega_i > 0$ (unstable modes).

CHAPTER 3

SOME GENERAL STABILITY CRITERIA

After Rayleigh's (1916) derivation of his condition for stability of rotating flows subject to axisymmetric disturbances using a conservation of angular momentum argument, Synge (1933) proved the same stability condition for heterogeneous fluids by casting the problem into the form of a Sturm-Liouville system and using the properties of this system. The resultant well-known Rayleigh-Synge criterion obtained is that the flow can be guaranteed stable for infinitesimal axisymmetric disturbances whenever the product of its density and the square of its circulation is an increasing function of radius.

We here give an alternate derivation of the Rayleigh-Synge criterion following Synge's approach but without assuming that the perturbations are periodic in the z direction.

For symmetry about the axis and zero axial flow, equations (2.5) can be reduced to

$$\frac{\partial \hat{p}}{\partial r} + \rho_o\left(\frac{\partial \hat{u}}{\partial t} - 2\Omega\hat{v}\right) - \hat{\rho} r\Omega^2 = 0 \qquad (3.1a)$$

$$\frac{\partial \hat{v}}{\partial t} + D^*(r\Omega)\hat{u} = 0 \quad (3.1b)$$

$$\frac{\partial \hat{p}}{\partial z} + \rho_o\left(\frac{\partial \hat{w}}{\partial t}\right) = 0 \quad (3.1c)$$

$$\frac{\partial \hat{\rho}}{\partial t} + (D\rho_o)\hat{u} = 0 \quad (3.1d)$$

$$\frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{\partial \hat{w}}{\partial z} = 0 \quad (3.1e)$$

Equations (3.1a) through (3.1d) can be combined into the following partial differential equation

$$\{\rho_o \frac{\partial^2}{\partial t^2} + \Phi\}\left(\frac{\partial \hat{u}}{\partial z}\right) - \frac{\partial^2}{\partial t^2}\{\frac{\partial}{\partial r}(\rho_o \hat{w})\} = 0 \quad (3.2)$$

Also we define a Stoke's stream function $\hat{\Psi}$ such that

$$\hat{u} = -\frac{1}{r}\frac{\partial \hat{\Psi}}{\partial z}$$
$$\hat{w} = \frac{1}{r}\frac{\partial \hat{\Psi}}{\partial r} \quad (3.3)$$

the continuity equation (3.1e) will be satisfied automatically, and (3.2) can be transformed to

$$\left\{\frac{\partial^2}{\partial t^2} + \frac{\Phi}{\rho_o}\right\} \frac{\partial^2 \hat{\Psi}}{\partial z^2} + \frac{\partial^2}{\partial t^2} \left\{\frac{r}{\rho_o} \frac{\partial}{\partial r}\left(\frac{\rho_o}{r} \frac{\partial \hat{\Psi}}{\partial r}\right)\right\} = 0 \tag{3.4}$$

Letting the infinitesimal perturbations vary exponentially in time (Yih 1965, Chapter 6) such that

$$\hat{\Psi} = \Psi(r,z)e^{-i\omega t}$$

allows us to reduce (3.4) to the form

$$\frac{\partial^2 \Psi}{\partial r^2} + \left[1 - \frac{1}{\omega^2}\left(\frac{\Phi}{\rho_o}\right)\right]\frac{\partial^2 \Psi}{\partial z^2} + \frac{D_*(\rho_o)}{\rho_o} \frac{\partial \Psi}{\partial r} = 0 \tag{3.5}$$

with boundary conditions $\frac{\partial \Psi}{\partial z} = 0$ at R_1 and R_2

The equation for characteristics of (3.5) is

$$\frac{dz}{dr} = \pm \sqrt{\frac{1}{\omega^2}\left(\frac{\Phi}{\rho_o}\right) - 1} \tag{3.6}$$

Equation 3.5 is hyperbolic if $\frac{1}{\omega^2}\left(\frac{\Phi}{\rho_o}\right) - 1 > 0$ (3.7a)

is parabolic if $\frac{1}{\omega^2}\left(\frac{\Phi}{\rho_o}\right) - 1 = 0$ (3.7b)

and is elliptic if $\frac{1}{\omega^2}\left(\frac{\Phi}{\rho_o}\right) - 1 < 0$ (3.7c)

There are two families of real characteristic curves describing the physical characteristic surfaces in the hyperbolic case, only one in the parabolic case and none in the elliptic case. For this reason, two conclusions can be drawn just by discussing the characteristics of the partial differential equation (3.5).

First, real characteristics do not exist if ω^2 is complex since $\frac{\Phi}{\rho_o}$ is always real. In order to have at least one set of real characteristic curves satisfy a physical situation, it requires ω^2 be real, i.e., the amplification factor is either real (stability) or pure imaginary (instability). Hence the principle of exchange stabilities holds for the flow under consideration with $\omega = 0$ corresponding to the neutral stability condition.

Second, the assumed stream function Ψ (r, z) is a scalar function such that

$$d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial z} dz = 0$$

or

$$\frac{dz}{dr} = - \frac{\frac{\partial \Psi}{\partial r}}{\frac{\partial \Psi}{\partial z}} \qquad (3.8)$$

Substituting (3.3) for exponential solutions in time, i.e.,

$$\{\hat{u}, \hat{w}\} = \{\bar{u}(r,z), \bar{w}(r,z)\} e^{-i\omega t}$$

in (3.8) yields

$$\frac{dz}{dr} = \frac{\bar{w}}{\bar{u}} \tag{3.9}$$

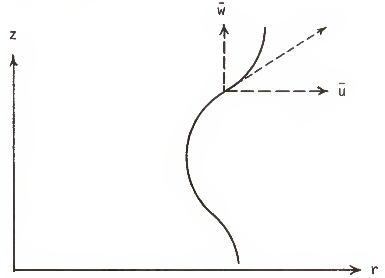


Figure 3.1 Fluid particle movements in the velocity field for axisymmetric disturbances.

From the definition of the stream function, the fluid particles should move along the streamlines described by the stream function with the slope governed by equation (3.9) (see Figure 3.1). In the case under consideration here the equation describing the slope of the stream function happens to be the equation for characteristics of the partial differential equation (3.5) governing stability. In order to have real streamlines as well as real fluctuation

velocities $\bar{u}$ and $\bar{w}$ exist, it is required that the slope of streamlines in (3.6) and (3.9) be real, i.e., equation (3.5) can either be hyperbolic or parabolic. Accordingly the criterion governing stability is

$$\frac{1}{\omega^2}\left(\frac{\Phi}{\rho_o}\right)-1 \geq 0 \qquad (3.10)$$

i.e., $\Phi \geq \rho_o\omega^2$ for positive ω^2, stability

and $\Phi < \rho_o\omega^2$ for negative ω^2, instability

The well-known Rayleigh-Synge criterion, $\Phi \geq o$, then follows from the principle of exchange stabilities with $\omega = 0$ corresponding to neutral stability.

Perhaps we should remember that this is the criterion for instability of the Rayleigh-Taylor type since the mechanism is based on the particle exchanging due to the centrifugal (or circulation) effects. It should be emphasized once more that this criterion applies only to axisymmetric perturbations with zero axial velocity (or constant axial flows). Once perturbations are allowed to be non-axisymmetric or the axial velocity gradients are taken into

consideration, new mechanisms for instability will arise and instabilities of the Kelvin-Helmholtz type can then take place.

The investigations concerning instabilities of the Kelvin-Helmholtz type for heterogeneous shear flows was first attempted by Taylor (1931) and Goldstein (1931). By discussing the complex zeros of the Bessel functions, they discovered that the Richardson number, defined as the ratio of the density gradient of the flow to its shearing rate, equal to 1/4 was a critical value for stability. (The governing stability equation was then proposed by Thorpe (1969) to be called the Taylor-Goldstein equation, in recognition of its first use by Taylor and Goldstein in determining the stability of certain stratified flows.) Later Miles (1961) considered a more general stratified flow and proved the same sufficiency condition by applying the method of Frobenius to the Taylor-Goldstein equation, with a restriction that the velocity gradient was nowhere equal to zero, which was removed by Howard (1961). In addition, Howard illuminated a way to the study of shear instabilities by proving the well-known semi-circle theorem, which has been broadly used in the theoretical investigations of parallel flows as well as swirling flows.

Howard and Gupta (1962) published an important paper concerning stability of swirling flows without density

variations, in which instabilities of the Kelvin-Helmholtz type were discussed. We now want to examine the general features of stability for heterogeneous swirling flows. Two types of instabilities are under consideration: "centrifugal instabilities" representing unstable situations caused by centrifugal forces; and "shear instabilities" corresponding to unstable solutions produced by shearing effects. The former is an instability in the Rayleigh-Taylor sense while the latter belongs to an instability of the Kelvin-Helmholtz type.

Following Howard (1961), Howard and Gupta (1962), we assume there exists an unstable mode ($\omega_i > 0$) and let $u = N^{\zeta}\Psi$, where ζ is a real constant. Substituting this in equation (2.11), one obtains

$$D[(\rho_o EN^{2\zeta})D^{*}\psi] + N^{2\zeta-2}\{-\rho_o k^2N^2 + N[F + \zeta D_{*}(\rho_o EDN)] + [G + \zeta(\zeta - 1)\rho_o E(DN)^2]\}\psi = 0 \qquad (3.11)$$

Multiplying (3.11) by $r\bar{\psi}$ (the complex conjugate of $r\psi$) and integrating over the flow region, we make use of the boundary conditions $\psi(R_1) = \psi(R_2) = 0$ to obtain the integral form of the stability equation

$$\int_{R_1}^{R_2} N^{2\zeta-2}\{-\rho_o N^2[E|D^{*}\psi|^2 + k^2|\psi|^2] + N[F + \zeta D_{*}(\rho_o EDN)]|\psi|^2 + [G+ \zeta(\zeta - 1)\rho_o E(DN)^2]|\psi|^2\}rdr = 0 \qquad (3.12)$$

Several stability criteria for the heterogeneous swirling flows under consideration can be derived from this integral representation. These are now obtained as follows:

3.1 Sufficiency Condition for Stability

Letting $\zeta = \frac{1}{2}$ in (3.12) leads to the integral

$$\int_{R_1}^{R_2} \{-\rho_o N[E|D^*\psi|^2 + k^2|\psi|^2] + \{[F + \frac{1}{2} D_*(\rho_o EDN)] + \frac{1}{N}[G - \frac{1}{4}\rho_o E(DN)^2]\}|\psi|^2\} \, rdr = 0 \qquad (3.13)$$

whose imaginary part is

$$\omega_i \int_{R_1}^{R_2} \{\rho_o[E|D^*\psi|^2 + k^2|\psi|^2] + [G - \frac{1}{4}\rho_o E(DN)^2]\frac{|\psi|^2}{|N|^2}\} rdr = 0 \qquad (3.14)$$

It can be seen from (3.14) that unstable solutions ($\omega_i > 0$) are not possible if the integrand is positive definite throughout the whole integration domain. Therefore the flow can be guaranteed stable if the function $\{G - \frac{1}{4}\rho_o E(DN^2)\}$ is positive everywhere inside the flow region, that is,

$$[\frac{\Phi}{\rho_o} - \frac{1}{4}(DW)^2] - 2s(DW)[\Omega + \frac{1}{4}rD\Omega] + s^2[r\Omega^2 D(\log \rho_o) - \frac{1}{4} r^2(D\Omega)^2] \geq 0 \tag{3.15}$$

This criterion was first obtained by Kurzweg (1969) and is reducible to Howard and Gupta's (1962) result for constant density.

An alternate derivation of condition (3.15) can also be obtained by considering a Frobenius solution to equation (2.11) following an approach used by Miles (1961) in a related study on stratified shear flows. The derivation is given as follows:

Letting $u = \frac{\phi}{r}$ allows one to transform equation (2.11) to the form

$$D^2\phi + \frac{D_*(\rho_o E)}{\rho_o E} D\phi + \frac{1}{\rho_o E}[-\rho_o k^2 + \frac{F}{N} + \frac{G}{N^2}]\phi = 0 \tag{3.16}$$

The critical point $r = r_c$ of this equation is defined by $N = 0$

or equivalently $\quad kW + m\Omega \Big|_{r = r_c} = \omega$

with a restriction $\left. DW + srD\Omega \right|_{r = r_c} \neq 0$ in (R_1, R_2)

It is a regular singularity for the differential equation, and leads to an indicial equation

$$K^2 - K + \bar{g}_c = 0$$

or

$$K = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \bar{g}_c} \qquad (3.17)$$

where

$$\bar{g}_c = \left. \frac{\Phi - 2s\rho_o\Omega(DW) + s^2 r\Omega^2(D\rho_o)}{\rho_o(DW + srD\Omega)^2} \right|_{r = r_c}$$

The two linearly independent solutions to equation (3.16) are in the form

$$\phi_{\pm}(r) = (r-r_c)^{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \bar{g}_c}} \sum_{j=1}^{\infty} C_j(r-r_c)^j \qquad (3.18)$$

where C_j is arbitrary constant. The exponent of the solutions will be complex whenever $\bar{g}_c > \frac{1}{4}$, a condition identical

with that given by (3.15). Perhaps, one should go through some additional mathematical arguments in deriving this result by considering the Reynolds stress as done by Miles (1961). However, it follows from the sufficiency condition proven by the integral technique that neither ϕ_+ nor ϕ_- may have complex zeros and still match the boundary conditions when $\bar{g}_c > \frac{1}{4}$. The upper bound on which possible unstable solutions can ever exist is thus $\bar{g}_c = \frac{1}{4}$.

We have just shown from the above analysis that the sufficiency condition for stability depends completely on the second order singularity of the differential equation (2.11). As a result of it, two phenomena concerning stability analysis can be seen immediately:

First, neither the neglect of the inertia effects of density variation nor the narrow gap analysis will change the second order singularity of the stability equation, so that the sufficiency condition will remain unchanged. (A similar argument can be applied to the familiar Taylor-Goldstein equation arising in the stability of stratified shear flows.)

Second, when Howard and Gupta (1962) made this investigation of homogeneous swirling flow stability they realized that their sufficiency condition for stability was always violated for sufficiently small axial wave numbers and no general stability criterion was obtainable.

In order to explain this situation, one should look back at the original differential equation governing stability. For constant density and $k = 0$, the second order singularity of equation (2.11) will be eliminated and the equation they derived becomes an equation of the Rayleigh type having no second order singularity. Another look at the Frobenius solution (3.18) shows that the exponents of the indicial equation (3.17) are always real and equal to an integer because of the absence of the second order singularity. The second linearly independent solution is then proportional to $\phi_+(r)\log(r-r_c)$ and the solution for the perturbation velocity described by (3.18) may possess a complex zero matching the boundary conditions. Under this situation, the possibility of non-trivial solutions for unstable modes always exists and no stabilities can then be guaranteed before solving the equation. For this reason, the sufficiency condition derived by Howard and Gupta fails to predict stabilities due to the disappearance of the second order singularity in their equation for sufficiently small k. (Based on the same argument, the Richardson number criterion in the Taylor-Goldstein equation becomes meaningless in Rayleigh's equation for homogeneous fluids since the Richardson number is identically equal to zero due to the absence of buoyancy effects, and no general

information can be obtained as a result of this condition.)

In fact, the physical meaning of the sufficiency condition which is in terms of the second order singularity for the equations governing stability represents, in some way, the potential stabilizing effects existing in the flow to suppress instability. This phenomenon can be explained as follows:

Consider two particles A and B with equal fluid volumes in a two-dimensional parallel velocity field shown in Figure 3.2.

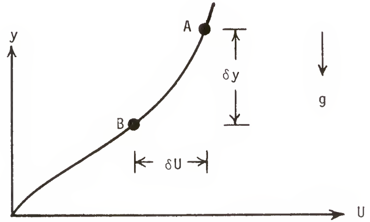


Figure 3.2 Fluid particle interchange in the gravitational force field.

Assume a statically stable profile with $\rho_A < \rho_B$ in a heterogeneous density field. The interchange of position between particle A and B is possible only when the

interchange kinetic energy gained by these two particles is sufficiently large to overcome the work done due to the exchange of them in the gravitational field. As a result of it, stability can be assured whenever the required work done exceeds the change in kinetic energy. In other words, instabilities are not possible when stabilizing effects are strong enough to suppress instabilities due to shear.

However, the interchange of the same two particles in a velocity field without gravitational effects or in a homogeneous density field with gravitational effects will result in no work done or potential energy changed. Due to the absence of stabilizing effects, any change in kinetic energy may cause instability. This is why the Richardson number criterion fails to predict stability for homogeneous fluids.

The same arguments can be used to explain the stability for non-rotating jets and for the homogeneous swirling flows against azimuthally periodic perturbations, with the centrifugal force field replacing the gravitational field. In both cases, no stabilizing factors exist and any axial or rotational shear component may upset the stability of the flow, so that no general conclusion for stability or possible unstable waves can be drawn. In the latter case, it suggests that the flow is less stable to two-dimensional azimuthally periodic modes (i.e., $k = 0$)

than the corresponding three-dimensional ones with non-zero axial wave numbers.

Condition (3.15) can be expressed in terms of a non-dimensional local quantity defined as

$$\lambda = \frac{(1 + s^2)\Phi}{\rho_0[DW + s(rD\Omega + 4\Omega)]^2} \qquad (3.19a)$$

If $DW + s(rD\Omega + 4\Omega) = 0$ somewhere, we allow $\lambda \to \infty$. In terms of this quotient the stability condition given by (3.15) is

$$\lambda \geq \frac{1}{4} \qquad (3.19b)$$

for all r in (R_1, R_2). The result obtained by Leibovich is deducible from this equation for the special case of axisymmetric perturbations (i.e., $s = 0$). Also the well-known Rayleigh-Synge criterion can be regained from (3.19b) in the absence of axial flow gradients and for $s = 0$.

This non-dimensional number λ, reminiscent of the Richardson number criterion for stability of heterogeneous shear flows, is defined as the ratio of centrifugal effects to shearing effects. Condition (3.19b) can be thought of as the criterion for shear stability in flows with "centrifugally stable" profiles. Here we say that a flow is "centrifugally stable" if the product of its density and

the square of its circulation is nowhere a decreasing function of radius. Condition (3.19b) clearly reveals that the centrifugal stability (i.e. $\Phi \geq 0$) does not necessarily lead to stability for nonaxisymmetric perturbations or in the presence of axial flow gradients. Accordingly it is concluded that the Rayleigh-Synge criterion, a condition for centrifugal stability, is neither necessary nor sufficient for shear stability.

It should be noted that Chandrasekhar in his book fails to realize this point and proposes to show that $\Phi \geq 0$ is a necessary and sufficient condition for the stability of swirling flows subject to arbitrary perturbations. As pointed out by Howard (1962) such a criterion is certainly incorrect.

Violations of the Rayleigh-Synge condition imply centrifugal instabilities for axisymmetric perturbations and, at the same time, result in the violation of the sufficiency condition (3.19b), which may induce shear instabilities (even though it is not necessarily so). Hence, two types of instabilities may be expected to take place simultaneously whenever $\Phi < 0$.

Here we wish to point out that equation (2.11) for the narrow gap approximation, i.e., $R_2 - R_1 << \frac{1}{2}(R_1 + R_2)$, in the absence of axial velocity gradients (DW = 0) is mathematically similar to the Taylor-Goldstein equation encountered in two-dimensional stratified shear flows.

Setting $\zeta = 1$ allows us to rewrite equation (3.11) as

$$D[(\rho_0 EN^2)D^*\psi] + \{-\rho_0 k^2 N^2 + N[F + D_*(\rho_0 EDN)] + G\}\psi = 0 \qquad (3.20)$$

Letting $r = R_0 + y$ and considering the small gap approximation $y << R_0$

so that
$$D^* \simeq D \simeq D_* = \frac{d}{dy}$$

with
$$s \simeq s_0 = \frac{m}{kR_0}$$

and
$$R_0 = \frac{1}{2}(R_1 + R_2)$$

allow us to reduce (3.20), after neglecting the axial velocity gradients, to a relatively simple form

$$\frac{d}{dy}[\rho_0(U - \hat{c})^2\frac{d\psi}{dy}] + [\hat{G} - \rho_0 k(1 + s_0)^2(U - \hat{c})]\psi = 0 \qquad (3.21)$$

with
$$U(y) = r\Omega$$
$$\hat{c} = \frac{1}{s_0}(\frac{\omega}{k} - W_0)$$
$$\hat{G} = \frac{1}{s_0^2}\frac{d}{dy}(\rho_0\frac{U^2}{R_0}) + \frac{d\rho_0}{dy}(\frac{U^2}{R_0})$$

If the centrifugal term $\frac{U^2}{R_o}$, produced by the steady-state rotation, has only small variations across the gap, then

$$\hat{G} \simeq (1 + \frac{1}{s_o^2})(\frac{U^2}{R_o})\frac{d\rho_o}{dy}$$

and equation (3.21) takes the form

$$\frac{d}{dy}[\rho_o(U - \hat{c})^2\frac{d\psi}{dy}] + [(1 + \frac{1}{s_o^2})(\frac{U^2}{R_o})\frac{d\rho_o}{dy} - \rho_o k^2(1 + s_o^2)(U - \hat{c})^2]\psi = 0 \quad (3.22)$$

This equation is mathematically identical to the Taylor-Goldstein equation before the effects of density variation in the inertia term are neglected, provided the centrifugal force term$\{\frac{U^2}{R_o}\}$ is replaced by the gravitational constant g. The sufficiency condition for stability of rotating flow in this narrow gap approximation is found to be

$$(1 + \frac{1}{s_o^2})J = [1 + (\frac{kR_o}{m})^2]J \geq \frac{1}{4} \quad (3.23)$$

with
$$J = \frac{-\frac{d\rho_o}{dy}(\frac{U^2}{R_o})}{\rho_o(\frac{dU}{dy})^2} \rightarrow \frac{-\rho_o' g}{\rho_o(U')^2}$$

being the well-known Richardson number encountered in heterogeneous parallel flows. Equation (3.23) will have

a minimum for each local "modified Richardson number" $[1 + (kR_0/m)^2]J$ or for each given velocity and density profile when k goes to zero, or m goes to infinity, implying that the flow is less stable with respect to two-dimensional azimuthally periodic perturbations than to the corresponding three-dimensional ones with $k \neq 0$. This in turn would imply that the Squire (1933) theorem (see also Yih 1955) for plane parallel flows with respect to three-dimensional perturbations would be expected in revolving flows as well, at least for large azimuthal wave numbers. This argument is supported by the analytical results by Michalke and Timme (1967) and by those of Fung and Kurzweg (1973), as well as experimental results by Weske and Rankin (1963) and by Johnson (1972) (see also Kurzweg 1974).

Following Taylor (1931) and Goldstein (1931), Drazin (1958) showed analytically that a quarter was the minimum value of the Richardson number J for which the flow can be guaranteed stable for stratified shear profiles. Later on Hazel (1972) confirmed this point by numerical investigations of the Taylor-Goldstein equation. Perhaps one may question if the sufficiency condition $\lambda \geq \frac{1}{4}$ for heterogeneous swirling flows can be improved. That this generally cannot be done is demonstrable by considering a homogeneous flow for rigid body rotation subject to azimuthally periodic disturbances. This flow is neutrally

stable (as will be shown through detailed calculation later) at precisely $\lambda = \frac{1}{4}$.

3.2 Necessary Condition for Instability

One of the necessary conditions for centrifugal instability subject to axisymmetric disturbances is well described by the Rayleigh-Synge criterion for zero axial velocity gradients. Two other necessary conditions for shear instability can be derived from the general integral equation (3.12) by letting $\zeta = 0$. This yields

$$\int_{R_1}^{R_2} \{-\rho_o[E|D^*\psi|^2 + k^2|\psi|^2] + [\frac{F}{N} + \frac{G}{N^2}]|\psi|^2\} rdr = 0 \qquad (3.24a)$$

whose real part is

$$\int_{R_1}^{R_2} \{F|N|^2(kW + m\Omega - \omega_r) + G[(kW + m\Omega - \omega_r)^2 - \omega_i^2]\}\frac{|\psi|^2}{|N|^4} rdr =$$

$$\int_{R_1}^{R_2} \rho_o[E|D^*\psi|^2 + k^2|\psi|^2] rdr \qquad (3.24b)$$

and imaginary part is

$$\omega_i \int_{R_1}^{R_2} \{F|N|^2 + 2G(kW + m\Omega - \omega_r)\} \frac{|\psi|^2}{|N|^4} r\,dr = 0 \qquad (3.24c)$$

We have not succeeded in drawing any general conclusion from (3.24c) for possible unstable solutions; however, some special cases in which the function {G} vanishes can be obtained. In such a case, exactly as in the proof of Rayleigh's (1880) inflection point theorem for instability of inviscid parallel flows, we can conclude that the function inside the curly bracket in (3.24c) should change sign in (R_1, R_2). Certain special cases of equations (3.24) can lead to the following results:

3.2.1 Non-axisymmetric Perturbations of Heterogeneous Axial Flows

Equations (3.24b,c) for $\Omega = 0$ yield

$$\int_{R_1}^{R_2} (W - c_r) D_*(\rho_0 EDW) \frac{|\psi|^2}{|W - c|^2} r\,dr = -\int_{R_1}^{R_2} \rho_0 [E|D^*\psi|^2 + k^2|\psi|^2] r\,dr$$

$$< 0 \qquad (3.25a)$$

$$c_i \int_{R_1}^{R_2} D_*(\rho_0 EDW) \frac{|\psi|^2}{|W - c|^2} r\,dr = 0 \qquad (3.25b)$$

where

$$c = \frac{\omega}{k} = c_r + ic_i$$

In order to have unstable solutions $(c_i > 0)$ we require that the quantity $\{\frac{\rho_0 rDW}{m^2 + (kr)^2}\}$ have at least one numerical extremum value at some point in (R_1, R_2). This is exactly the condition Batchelor and Gill (1962) derived except that they only studied the homogeneous jets (see also Rayleigh 1892a). However, the density variation in no way influences the characteristics of the general criteria (see both the necessary condition and the later to be discussed semi-circle theorem) for the pure axial flows if the gravitational effects are neglected.

An extension of Rayleigh's theorem by Fjørtoft (1950) and Høiland (1953) states that the necessary condition for instability is that the vorticity should somewhere have a maximum inside the flow domain. A similar refinement of the above necessary condition for instability of radially dependent jet flows can be also made. Let r_s be the point where the function $\{D_*(\rho_0 EDW)\}$ vanishes, the corresponding axial velocity being W_s. Equations (3.25b) can be written as

$$(W_s - c_r)\int_{R_1}^{R_2} D_*(\rho_0 EDW)\frac{|\psi|^2}{|W - c|^2} rdr = 0 \qquad (3.25c)$$

Equations (3.25a,c) then yield

$$\int_{R_1}^{R_2} (W - W_s)D_*(\rho_o EDW)\frac{|\psi|^2}{|W - c|^2} rdr = -\int_{R_1}^{R_2} \rho_o[E|D^*\psi|^2 + k^2|\psi|^2]rdr < 0$$

From this follows a stronger necessary condition for instability, namely that $(W - W_s)D_*(\rho_o EDW) < 0$ somewhere in the flow field. In particular, if $W(r)$ is a monotone function and $D_*(\rho_o EDW)$ vanishes only once in (R_1, R_2), a necessary condition for instability is that $(W - W_s)D(\frac{\rho_o rDW}{m^2 + k^2r^2}) \le 0$ through the flow region.

3.2.2 <u>Azimuthally Periodic Perturbations of Homogeneous Swirling Flows</u>

For $k = 0$ and $D\rho_o = 0$, the sufficiency condition (3.19b) fails to predict stability because of the absence of centrifugal stabilizing effects. The necessary condition for instability is that $DD^*(r\Omega)$ should change sign, i.e., the vorticity in axial direction should have a maximum in (R_1, R_2). This result is due to Rayleigh (1880) for zero axial velocity; the presence of axial flows of course has no relevant effects when the perturbations are azimuthally periodic. Unquestionably this necessary condition can also be refined using Fjørtoft and Høiland's method.

For instability, the term $\{(\Omega - \Omega_s)D^*(r\Omega)\}$ should be negative somewhere in the field of flow, Ω_s being the value of the angular velocity where the axial vorticity component vanishes.

3.3 Bound for the Complex Wave Speed

A general bound on the growth rate for centrifugal and/or shear instability can also derived from the general integral equation (3.12). For $\zeta = 1$, equation (3.12) has the complex form

$$\int_{R_1}^{R_2} \{N^2\rho_o[E|D^*\psi|^2 + k^2|\psi|^2] - \{N[F + D_*(\rho_o EDN)] + G\}|\psi|^2\}rdr = 0 \tag{3.26}$$

with the real part

$$\int_{R_1}^{R_2} \{[(kW + m\Omega - \omega_r)^2 - \omega_i^2]Q$$

$$- \{(kW + m\Omega - \omega_r)[F + D_*(\rho_o EDN)] + G\}|\psi|^2 r\}dr = 0 \quad (3.27a)$$

and imaginary part

$$\int_{R_1}^{R_2} (kW + m\Omega)Q dr = \omega_r \int_{R_1}^{R_2} Q dr + \frac{1}{2}\int_{R_1}^{R_2} [F + D_*(\rho_0 EDN)]|\psi|^2 r dr \qquad (3.27b)$$

where

$$Q = \rho_0[E|D^*\psi|^2 + k^2|\psi|^2]r > 0$$

Substituting (3.27b) to (3.27a), we obtain

$$\int_{R_1}^{R_2} (kW + m\Omega)^2 Q dr = (\omega_r^2 + \omega_i^2)\int_{R_1}^{R_2} Q dr + \int_{R_1}^{R_2} \{(kW + m\Omega)$$

$$\cdot[F + D_*(\rho_0 EDN)] + G\}\ |\psi|^2 r dr \qquad (3.27c)$$

The above integrals are reminiscent of those derived by Howard (1961), and Howard and Gupta (1962) in their formulation of the semi-circle theorem serving as the bound of the complex velocity in stratified shear flows.

Two types of semi-circle bound for instability of swirling flows will now be discussed.

3.3.1 The Semi-circle Theorem for Shear Instabilities

For the present swirling flows, the semi-circle theorem can also be derived for special profiles. We consider the following four categories:

(a) For non-rotating jet flows, $\Omega = 0$, equations (3.27b,c) become

$$\int_{R_1}^{R_2} WQdr = c_r \int_{R_1}^{R_2} Qdr \tag{3.28a}$$

$$\int_{R_1}^{R_2} W^2 Qdr = (c_r^2 + c_i^2) \int_{R_1}^{R_2} Qdr \tag{3.28b}$$

where $c = \frac{\omega}{k}$, the complex eigen wave speed.

Suppose that $a \le W \le b$, a and b being the lower and upper bound of the axial velocity respectively. Equations (3.28a, b) imply

$$0 \le \int_{R_1}^{R_2} (W - a)(W - b)Qdr = \int_{R_1}^{R_2} W^2 Qdr - (a + b) \int_{R_1}^{R_2} WQdr + ab \int_{R_1}^{R_2} Qdr$$

$$= \{c_r^2 - (a + b)c_r + c_i^2 + ab\} \int_{R_1}^{R_2} Qdr$$

It follows that

$$\{c_r - \tfrac{1}{2}(a + b)\}^2 + c_i^2 \le \{\tfrac{1}{2}(a - b)\}^2 \tag{3.29}$$

This is equivalent to Howard's semi-circle theorem for stratified shear flows, saying that the complex wave speed for unstable modes must lie within a certain semi-circle, with the range of the axial velocity for diameter. It was Batchelor and Gill (1962) who first looked at the semi-circle theorem for homogeneous non-rotating jet flows. From our result it is clear theirs can be extended to flows with radial density variations.

(b) For axisymmetric perturbations (m = 0), equations (3.27b.c) reduce to

$$\int_{R_1}^{R_2} WQdr = c_r \int_{R_1}^{R_2} Qdr \tag{3.30a}$$

$$\int_{R_1}^{R_2} W^2 Qdr = (c_r^2 + c_i^2) \int_{R_1}^{R_2} Qdr + \int_{R_1}^{R_2} \Phi|\psi|^2 rdr \tag{3.30b}$$

If $a \leq W \leq b$, then equations (3.30) imply

$$0 \leq \int_{R_1}^{R_2} (W - a)(W - b)Qdr$$

$$= \{c_r^2 - (a + b)c_r + c_i^2 + ab\} \int_{R_1}^{R_2} Qdr + \int_{R_1}^{R_2} \Phi|\psi|^2 rdr$$

Consider only centrifugally stable profiles, i.e., $\Phi \geq 0$. One concludes that

$$\{c_r - \tfrac{1}{2}(a + b)\}^2 + \{c_i\}^2 \le \{\tfrac{1}{2}(a - b)\}^2 \qquad (3.31)$$

From examples in Sections 5.1.1(b) and 5.2.3, the existence of this semi-circle theorem, first derived by Leibovich (1969) and reducible to Howard and Gupta's result for constant density, suggests that even for axisymmetric perturbations shear instabilities may still exist when the Rayleigh-Synge condition is satisfied, i.e., $\Phi \geq 0$.

(c) For azimuthally periodic perturbations ($k = 0$) and $D(\rho_0\Omega) = 0$, equations (3.27b,c) reduce to

$$\int_{R_1}^{R_2} \Omega\hat{Q}dr = \hat{c}_r \int_{R_1}^{R_2} \hat{Q}dr \qquad (3.32a)$$

$$\int_{R_1}^{R_2} \Omega^2\hat{Q}dr = (\hat{c}_r^2 + \hat{c}_i^2) \int_{R_1}^{R_2} \hat{Q}dr + \int_{R_1}^{R_2} r\Omega^2(D\rho_0)\frac{|\psi|^2}{m^2}rdr \qquad (3.32b)$$

where

$$\hat{Q} = \rho_0[\frac{r^2}{m^2}|D^*\psi|^2 + |\psi|^2]r > 0$$

$$\hat{c} = \frac{\omega}{m}$$

Let a and b be the upper and lower bound of the angular velocity Ω, and making use of equations (3.32) we obtain

$$0 \le \int_{R_1}^{R_2} (\Omega - a)(\Omega - b)\hat{Q}dr$$

$$= \{\hat{c}_r^2 - (a + b)\hat{c}_r + \hat{c}_i^2 + ab\} \int_{R_1}^{R_2} \hat{Q}dr + \int_{R_1}^{R_2} r\Omega^2(D\rho_o)\frac{|\psi|^2}{m^2}rdr$$

For centrifugally stable profiles with $D\rho_o \ge 0$ we have

$$\{\hat{c}_r - \tfrac{1}{2}(a + b)\}^2 + \{\hat{c}_i\}^2 \le \{\tfrac{1}{2}(a - b)\}^2 \qquad (3.33)$$

Unfortunately the requirement that $D(\rho_o\Omega) = 0$ makes this result lack generality.

(d) For the narrow gap approximation, equations (3.27b,c) take the form

$$\int_{R_1}^{R_2} (W + s_o r\Omega)Qdr = c_r \int_{R_1}^{R_2} Qdr \qquad (3.34a)$$

$$\int_{R_1}^{R_2} (W + s_o r\Omega)Qdr = (c_r^2 + c_i^2) \int_{R_1}^{R_2} Qdr + \int_{R_1}^{R_2} G\frac{|\psi|^2}{k^2}rdr \qquad (3.34b)$$

Here $s \simeq s_o = \frac{m}{kR_o}$ is assumed to be finite.

Letting

$$a \leq W + s_o r\Omega \leq b$$

then

$$0 \leq \int_{R_2}^{R_1} [(W + s_o r\Omega) - a][(W + s_o r\Omega) - b]Q\,dr$$

$$= \{c_r^2 - (a + b)c_r + c_r^2 + ab\} \int_{R_1}^{R_2} Q\,dr + \int_{R_1}^{R_2} G\frac{|\psi|^2}{k^2} r\,dr$$

For centrifugally stable profiles where $G \geq 0$, it requires

$$\{c_r - \tfrac{1}{2}(a + b)\}^2 + c_i^2 \leq \{\tfrac{1}{2}(a - b)\}^2 \qquad (3.35)$$

The semi-circle theorem here states that the complex eigen axial wave speed $c = \frac{\omega}{k}$ for unstable solutions under narrow gap approximation must lie inside a semi-circle which has the range of the "swirling velocity" $(W + s_o r\Omega)$ as diameter.

Two phenomena can be observed from the derivation of the semi-circle theorem for the four special profiles (and for two-dimensional stratified shear flows as well) mentioned above. First, it appears that the semi-circle theorem yields a bound upon the complex wave velocity (either c or $\hat{c}$) only for shear instabilities with centrifugally stable profiles; second, this velocity-dependent bound is explicitly independent of the boundary or density effects.

3.3.2 The Semi-circle Bound for Rigid Body Rotation

The semi-circle theorem we have just discussed is the bound of complex wave speeds for centrifugally stable profiles encountered in shear motion. A similar criterion can also be found for instability of rigid body rotation subject to azimuthally periodic disturbances.

Considering $k = 0$ and $\Omega = \Omega_o$ in equation (2.11), one obtains

$$D(\rho_o r D\phi) - [\frac{\rho_o m^2}{r} + \Lambda(D\rho_o)]\phi = 0 \tag{3.36}$$

with the boundary conditions

$$\phi(R_1) = \phi(R_2) = 0$$

where

$$\phi = ru$$

$$\Lambda = \frac{2}{1 - \frac{\hat{c}}{\Omega_o}} - \frac{1}{(1 - \frac{\hat{c}}{\Omega_o})^2} \tag{3.37a}$$

with the imaginary part

$$\Lambda_i = \frac{2(\frac{\hat{c}_i}{\Omega_o})}{|1 - \frac{\hat{c}}{\Omega_o}|^4} [(\frac{\hat{c}_r}{\Omega_o} - \frac{1}{2})^2 + (\frac{\hat{c}_i}{\Omega_o})^2 - \frac{1}{4}] \tag{3.37b}$$

$$\hat{c} = \frac{\omega}{m} = \hat{c}_r + i\hat{c}_i$$

Equation (3.36) with boundary conditions forms a Sturm-Liouville system which requires that: (1) Λ be real; (2) Λ and $D\rho_0$ be of opposite sign. The first condition for $\hat{c}_i > 0$ implies

$$(\hat{c}_r - \frac{\Omega_0}{2})^2 + \hat{c}_i^2 = (\frac{\Omega_0}{2})^2 \tag{3.38}$$

The complex azimuthal wave velocity for unstable modes must lie exactly on a semi-circle with the rotating velocity for diameter. Meanwhile (3.37a) can be rearranged as

$$\frac{\hat{c}}{\Omega_0} = \frac{\Lambda - 1 \pm \sqrt{1 - \Lambda}}{\Lambda} \tag{3.39}$$

$\hat{c}$ will be complex whenever $\Lambda > 1$, so that $D\rho_0 < 0$ for instability followed from the second condition of the Sturm-Liouville system. Note that if $D\rho_0 \geq 0$ then $\Lambda < 0$ which implies stability. This sufficiency condition for stability is also obtainable from condition (3.15) by considering rigid body rotation for azimuthal periodic disturbances.

3.4 <u>The Growth Rate</u>

The semi-circle bounds we have discussed are the bounds of the complex wave speeds for particular types of velocity and density profiles under the influence of shear

instability. A general upper bound on the growth rates for both centrifugal and shear instabilities can also be derived from the general integral equation (3.12).

Equation (3.24c) for possible instability requires

$$F|N|^2 + 2G(kW + m\Omega - \omega_r) = 0 \qquad (3.40)$$

at least at one point in (R_1, R_2). Equation (3.40) is a quadratic equation for the complex ω and can be rearranged into

$$kW + m\Omega - \omega_r = -G \pm \sqrt{G^2 - F^2\omega_i^2} \qquad (3.41)$$

The left-hand side of (3.41) is a real quantity, and it follows that the function inside the radical should be positive to have real solutions, i.e.,

$$G^2 - F^2\omega_i^2 \geq 0$$

or

$$\omega_i \leq \max\left|\frac{G}{F}\right| \qquad (3.42)$$

provided $\{F\}$ does not change sign in (R_1, R_2).

We can observe immediately from (3.42) that this bound fails to predict growth rates when $\{F\}$ becomes

sufficiently small anywhere inside the flow domain. For instance, ω_i becomes infinity for a potential vortex with a superposed constant axial flow, which is actually stable against all disturbances except for negative density gradients.

On the other hand, equation (3.42) can reasonably predict the growth rate for solid body rotation subject to azimuthally periodic perturbations. The upper bound of the present flow predicted in this case is $\omega_i \leq |\frac{m}{2}\Omega_o|$, which is exactly equivalent to the result obtained from (3.38).

Another second upper bound on the growth rate is obtainable from the general integral equation by setting $\zeta = \frac{1}{2}$. For instability, equation (3.14) can be written as

$$\int_{R_1}^{R_2} \{\rho_o k^2|N|^2 + [G - \tfrac{1}{4}\rho_o E(DN)^2]\}\frac{|\psi|^2}{|N|^2} r dr = -\int_{R_1}^{R_2} \rho_o E|D^*\psi|^2 r dr \leq 0 \tag{3.43}$$

The inequality

$$\rho_o k^2|N|^2 + [G - \tfrac{1}{4}\rho_o E(DN)^2] \leq 0$$

must hold somewhere inside the region of the flow. Since $\omega_i^2 \leq |N|^2$, we find

$$\omega_i^2 \leq \max\{\frac{1}{4}\,\frac{[DW + s(4\Omega + rD\Omega)]^2}{1 + s^2} - \frac{\Phi}{\rho_o}\} \tag{3.44}$$

This is reducible to Howard and Gupta's result for constant density. Several observations can be made from this upper bound immediately. Axial and angular velocity gradients destabilize the flow and definitely increase the growth rate, while the centrifugal effects stabilize the flow and reduce the growth rate. No instabilities can occur whenever the centrifugal stabilizing effects are sufficiently strong to suppress the destabilizing ones of shearing rates of both the axial and tangential velocity components.

CHAPTER 4

INTERFACIAL CONDITIONS

From the general criteria discussed in Chapter 3, stable swirling flows will exist whenever $\lambda \geq \frac{1}{4}$ is satisfied. However, violations of this sufficiency condition do not necessarily imply instability. In order to determine if the flow is unstable when condition (3.19b) is violated, it is necessary to solve the second order differential equation (2.11) or the two corresponding first order differential equation (2.9) and (2.10) exactly. Unfortunately, examples of swirling flows for which the stability problem can be explicitly solved do not seem to be readily found. The trouble with finding analytical solutions to (2.11) is that the equation possesses a second order as well as a first order singularity. Equation (2.11) is singular whenever $N = kW + m\Omega - \omega = 0$ and only series solutions can be obtained by the Frobenius method with a restriction that the swirling velocity gradient $\{DW + srD\Omega\}$ should nowhere vanish inside the flow region. A second look at the singularity shows the difficulty in locating the possible singular points which

are in terms of the unknown eigen wave speed.

Since explicit solutions to the eigenvalue problem, especially for unstable modes, are difficult to find for smoothly varying functions, the use of non-smoothly varying profiles can be sometimes justified as an approximation to related smoothly varying ones. The analysis of the instability for broken-line velocity and density distributions has been extensively covered in the literature and can be employed to yield many instability characteristics. In general, this analysis will allow simple solutions to the governing differential equation in the individual region which possesses continuous simple-varying velocity and density distributions. We will proceed to construct the stability characteristics of such broken-line velocity and density profiles.

In deriving the stability equation, we have not considered the possibility of the existence of forces (such as surface tension and centrifugal effects) arising from the possible discontinuity at the interfaces. Modifications in this case become necessary due to the important roles played by these forces which do not exist in continuous velocity and density distributions.

Consider surface tension T_j acting at the interface j located between two rings of fluids at $r = R_j$. The equation of motion for the mean flow in the radial

direction is

$$\frac{dP_o}{dr} - \rho_o r\Omega^2 = - \sum_{j=1}^{M} T_j \left(\frac{1}{R_j}\right) \delta(r - R_j) \tag{4.1}$$

where $\delta(r - R_j)$ is a Delta function, R_j is the position as well as the principal radius of curvature for the disturbed interface.

Once the flow is disturbed, the principal radii of curvature on $r - \theta$ and $r - z$ planes are specified as

$$\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}} = \frac{1}{r} - \frac{1}{r^2}\frac{\partial^2 r}{\partial \theta^2} - \frac{\partial^2 r}{\partial z^2} \tag{4.2}$$

and the linearized perturbation equation of motion in the radial direction yields

$$\frac{dP_o}{dr} + \frac{\partial \hat{p}}{\partial r} + \rho_o(d_* \hat{u} - 2\Omega\hat{v}) - (\rho_o + \hat{\rho})r\Omega^2 =$$

$$- \sum_{j=1}^{M} T_j\left(\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}}\right) \delta(r - R_j) \tag{4.3}$$

Subtracting (4.1) from (4.3), one finds

$$\frac{\partial \hat{p}}{\partial r} + \rho_o \ (d_* \hat{u} - 2\Omega \hat{v}) - \hat{\rho} r \Omega^2 = \sum_{j=1}^{M} T_j [\frac{1}{R_j} - (\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}})] \ \delta(r - R_j) \qquad (4.4)$$

In the absence of surface tension, equation (4.4) becomes identically equal to equation (2.5a). The validity of the equations of motion in the θ and z directions, the continuity equation and the incompressibility condition is unaffected by the existence of surface tension and needs no modification.

Following the normal mode analysis and assuming infinitesimal disturbances, we let

$$r = R_j + \hat{\eta} = R_j + \eta(R_j) e^{i(kz + m\theta - \omega t)}$$

at the interface j. Equation (4.2) can then be linearized into the form

$$\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}} = \frac{1}{R_j} - \frac{1}{R_j^2} (1 - m^2 - \kappa_j^2) \ \eta_j e^{i(kz + m\theta - \omega t)} \qquad (4.5)$$

where

$$\eta_j = \eta(R_j) = -i(\frac{u}{N})_j$$

$$\kappa_j = kR_j$$

Assume periodic functions for all the perturbation quantities, i.e.,

$$\{\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\rho}\} = \{u(r), v(r), w(r), p(r), \rho(r)\}e^{i(kz + m\theta - \omega t)} \tag{4.6}$$

Equations (4.4), (4.5) and (2.5b to e) can be transformed into ordinary differential equations, namely,

$$Dp + \rho_o[iNu - 2\Omega v] - \rho r\Omega^2 = \sum_{j=1}^{M} \frac{T_j}{R_j^2}(1 - m^2 - \kappa_j^2)\eta_j\delta(r - R_j)$$

$$p + \rho_o \frac{r}{m}[Nv - iD^*(r\Omega)u] = 0$$

$$p + \frac{\rho_o}{k}[Nw - i(DW)u] = 0 \tag{4.7}$$

$$iN\rho + (D\rho_o)u = 0$$

$$D^*u + ik(sv + w) = 0$$

Two first order differential equations can be deduced from (4.7).

$$(k^2 + \frac{m^2}{r^2})p = i\rho_o[k(DW) + \frac{mD^*(r\Omega)}{r} - D^*]u \tag{4.8a}$$

$$NDp + \frac{2m\Omega}{r}p = i\{(\Phi - \rho_o N^2)u - N\sum_{j=1}^{M}\frac{T_j}{R_j^2}(1 - m^2 - \kappa_j^2) \cdot \left(\frac{u}{N}\right)_j \delta(r - R_j)\} \tag{4.8b}$$

Equations (4.8) are exactly equivalent to (2.9) and (2.10) except for the surface tension term. The modifications of the stability equation due to surface tension arising at interfaces result in

$$N^2[D(\rho_o ED^*) - \rho_o k^2]u + [NF + G]u = K^2 N \sum_{j=1}^{M}\frac{T_j}{R_j^2}(1 - m^2 - \kappa_j^2)\left(\frac{u}{N}\right)_j \delta(r - R_j) \tag{4.9}$$

Equation (4.9) without surface tension reduces to (2.11), which is valid at points not on a discontinuity surface. Alterman's (1961) result can be deduced for axisymmetrical disturbances.

Both the displacement and pressure conditions will be discussed mathematically and physically as follows:

4.1 Kinematic Interfacial Condition

Consider two rings of fluids in region 1 and region 2 with an interface at $r = R$. Integrating (4.8a) from $R - \varepsilon$ to $R + \varepsilon$ and assuming that the mean value exists

for all the quantities of the flow and that their derivatives should be bounded at the interface, we obtain

$$\lim_{\varepsilon \to 0} \left. \frac{u}{N} \right|_R \int_{R-\varepsilon}^{R+\varepsilon} [m(D\Omega) + k(DW)]dr - \int_{R-\varepsilon}^{R+\varepsilon} (Du)dr = 0$$

or

$$\left. \frac{u}{N} \right|_R [m(\Omega_2 - \Omega_1) + k(W_2 - W_1)] - [u_2 - u_1] = 0 \qquad (4.10)$$

where

$$\left. \frac{u}{N} \right|_R = \frac{1}{2}\left(\frac{u_1}{N_1} + \frac{u_2}{N_2}\right),$$

the mean value at the interface. Define

$$\langle f \rangle = f(R_{+o}) - f(R_{-o}) \qquad (4.11)$$

a jump which a quantity $\{f\}$ experiences at the interface $r = R$. Without making use of any physical arguments, equation (4.10) can be written as

$$\langle \frac{u}{N} \rangle = 0 \qquad (4.12)$$

the quantity $\{\frac{u}{N}\}$, physically interpreted as a perturbation displacement, must be nowhere discontinuous at the interface

or anywhere inside the domain of the flow.

Condition (4.12) is also obtainable by the following physical argument:

Let the radial perturbation displacement be $\hat{\eta}$. The analysis of the normal mode method allows all the perturbation quantities to admit solutions with periodic behavior in z, θ and t and with radially dependent amplitudes. Hence we assume

$$\hat{\eta}(r, \theta, z; t) = \eta(r)e^{i(kz + m\theta - \omega t)} \tag{4.13}$$

According to the chain rule

$$\begin{aligned} \frac{d\hat{\eta}}{dt} &= \frac{\partial\hat{\eta}}{\partial t} + \frac{\partial\hat{\eta}}{\partial r}\frac{dr}{dt} + \frac{\partial\hat{\eta}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{\eta}}{\partial z}\frac{dz}{dt} \\ &= i(-\omega + m\Omega + kW)\eta(r)e^{i(kz + m\theta - \omega t)} \qquad (4.14) \\ &= iN\eta(r)e^{i(kz + m\theta - \omega t)} \end{aligned}$$

By definition, the total derivative of the radial perturbation displacement is equal to the radial perturbation velocity which has the form

$$\hat{u} = u(r)e^{i(kz + m\theta - \omega t)} \tag{4.15}$$

Equating (4.14) and (4.15), one finds,

$$\eta = \frac{u}{iN} \tag{4.16}$$

Equation (4.12) then follows from the physical argument that the normal component of the perturbation displacement must be continuous across the material fluid interface inside the flow domain.

4.2 Dynamic Interfacial Condition

The dynamic boundary condition at an interface between two fluids has been carried out through different arguments for continuous and discontinuous velocity and density distributions. The application of an integration technique to equations governing stability across interfaces has been used to lead to explicit interfacial conditions for problems in three-dimensional parallel flows (Chandrasekhar 1961) and two-dimensional parallel flows (Drazin and Howard 1966). Alterman (1961) looked at the stability characteristics of a heterogeneous liquid with radially dependent azimuthal and axial velocities. For antisymmetric disturbances, he obtained an interfacial condition for two-region flows by integrating the governing second order differential equation across the interface between the two fluids. Kent *et al*. (1969) used the

same method to obtain a jump condition at the interface for the Kelvin-Helmholtz instability around an isothermal, cylindrical plasma in an arbitrary radially dependent density and electric field. Unfortunately, this application of the integration technique to second order equations governing stability can be used to lead to simple results for only some particular cases. General pressure conditions cannot easily be found by this approach. However, the general dynamic interfacial condition can in fact be derived, using the integration technique across the interfaces, from one of the two first order differential equations leading to the stability equation. Equation (4.8b) can be rearranged to the form

$$Dp + \frac{2m\Omega}{Nr}p = i\{[D(\rho_o r\Omega^2) + \rho_o(3\Omega^2 - N^2)](\frac{u}{N})$$

$$- \sum_{j=1}^{M} \frac{T_j}{R_j^2}(1 - m^2 - \kappa_j^2)(\frac{u}{N})_j \delta(r - R_j)\} \qquad (4.17)$$

Applying the integration technique across the interface at r = R, we obtain

$$\lim_{\varepsilon \to 0} \int_{R-\varepsilon}^{R+\varepsilon} \{Dp - i[D(\rho_o r\Omega^2)(\frac{u}{N}) - \frac{T}{R^2}(1 - m^2 - \kappa^2)$$

$$\cdot(\frac{u}{N})_R \delta(r - R)]\} \, dr = 0$$

or

$$\langle p \rangle - i\left(\frac{u}{N}\right)_R \{\langle \rho_o r\Omega^2 \rangle - \frac{T}{R^2}(1 - m^2 - \kappa^2)\} = 0 \qquad (4.18)$$

$\langle\ \rangle$, defined in (4.11), represents a possible jump condition at the interface.

The reason why the general dynamic interfacial condition is difficult to obtain by integrating the second order differential equation for stability has now been revealed. Equations (4.8a,b), the two first order differential equations leading to the stability equation, represent the kinemetic and dynamic interfacial conditions respectively, which couple together in the second order stability equation. Except in special cases they are not likely, in general, to be separated if integration techniques are considered only with respect to the equation governing stability.

Equation (4.18) may also be reached through the following physical argument:

Consider two regions of fluids with an interface at $r = R$ inside an annular ring $R_1 \leq r \leq R_2$. The equation of motion (4.1) is valid for the flow in both regions. Let the subscripts 1 and 2 represent the quantities in region 1 and region 2 respectively. Integrate (4.1) from the inner boundary outwards for both regions. The total

pressures for the steady-state are

$$P_1(r) = \int_{R_1}^{R} \rho_1 r\Omega^2 dr \qquad \text{for } R_1 \le r \le R \tag{4.19a}$$

$$P_2(r) = \lim_{\varepsilon\to 0} \{ \int_{R_1}^{R-\varepsilon} \rho_1 r\Omega_1^2 dr - \int_{R-\varepsilon}^{R+\varepsilon} \frac{T}{R^2}\delta(r - R)dr + \int_{R+\varepsilon}^{r} \rho_2 r\Omega_2^2 dr\}$$

$$= \int_{R_1}^{R} \rho_1 r\Omega_1^2 dr + \int_{R}^{r} \rho_2 r\Omega_2^2 dr - \frac{T}{R^2} \quad \text{for } R \le r \le R_2 \tag{4.19b}$$

where T stands for the surface tension evaluated at the interface. From (2.4) and (4.19), the total pressures for the disturbed state evaluated at the deformed interface with the perturbation displacement $\eta(r, \theta, z; t)$ can be expressed as

$$P_1(R + \hat{\eta}) = \int_{R_1}^{R+\hat{\eta}} \rho_1 r\Omega_1^2 dr + \hat{p}_1(R + \hat{\eta}) \tag{4.20a}$$

$$P_2(R + \hat{\eta}) = \int_{R_1}^{R} \rho_1 r\Omega_1^2 dr + \int_{R}^{R+\hat{\eta}} \rho_2 r\Omega_2^2 dr$$

$$- \frac{T}{R^2} + \hat{p}_2(R + \hat{\eta}) \tag{4.20b}$$

In the perturbed state, the total pressures evaluated at the deformed interface should be balanced by surface tension through the vortex sheet (see Figure 4.1).

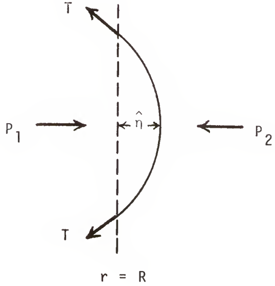


Figure 4.1 Total forces acting at the deformed interface.

$$P_1(R + \hat{\eta}) = P_2(R + \hat{\eta}) + T\left(\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}}\right) \tag{4.21}$$

or

$$\int_R^{R+\hat{\eta}} \rho_1 r\Omega_1^2 dr + \hat{p}_1(R + \hat{\eta}) = \int_R^{R+\hat{\eta}} \rho_2 r\Omega_2^2 dr + \hat{p}_2(R + \hat{\eta})$$

$$- T\left[\frac{1}{R} - \left(\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}}\right)\right] \tag{4.22}$$

Since both velocity and density profiles are bounded and continuous in $[R, R + \hat{\eta}]$ and the displacement $\hat{\eta}$ is assumed to be small, equation (4.22) can be written as

$$(\hat{p}_2 - \hat{p}_1)_R - T[\frac{1}{R} - \frac{1}{R_{r\theta}} + \frac{1}{R_{rz}})] = (\rho_1\Omega_1^2 - \rho_2\Omega_2^2)_R \int_R^{R+\hat{\eta}} r dr$$

$$= (\rho_1\Omega_1^2 - \rho_2\Omega_2^2)_R \{\frac{1}{2}[(R + \hat{\eta})^2 - R^2]\}$$

$$\simeq [(\rho_1\Omega_1^2 - \rho_2\Omega_2^2)R\hat{\eta}]_R \qquad (4.23)$$

Consider (4.5), (4.6) and (4.13) for periodic solutions. Equation (4.23) can then be simplified to

$$<p> + \{<\rho_o r\Omega^2> - \frac{T}{R^2}(1 - m^2 - \kappa^2)\} \eta_R = o \qquad (4.24)$$

Equation (4.18) follows by substituting (4.16) into the foregoing equation.

The physical meanings of the mathematical steps from (4.21) to (4.24) are in fact the dissolution of the total pressure forces acting at the perturbed interface into the force components acting at the steady-state interface, provided the perturbation quantities for the second or higher order are negligible. This procedure can be summarized in the following sketch:

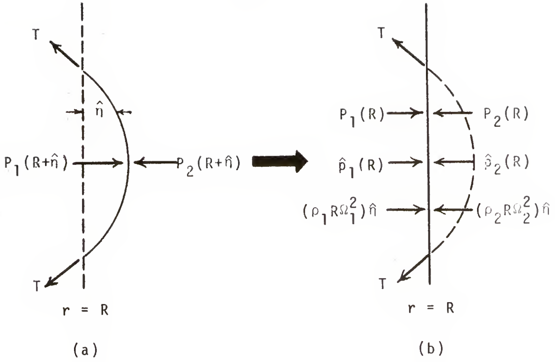


Figure 4.2 Dissolution of total pressure forces into force components at the interface.

In Figure 4.2b, we consider a differential area $dA = Rd\theta$ at the interface $r = R$. The centrifugal forces due to the rotation of the corresponding differential fluid volume $\hat{\eta}(Rd\theta)$ with respect to the axis are $(\rho_1 R\Omega_1^2)(\hat{\eta}Rd\theta)$ and $(\rho_2 R\Omega_2^2)(\hat{\eta}Rd\theta)$ for regions 1 and 2 respectively. Summing up all the radial forces acting at the fluid element, one obtains the equilibrium condition

$$(P_1 + \hat{p}_1)(Rd\theta) + \rho_1 R\Omega_1^2(\hat{\eta}Rd\theta) = (P_2 + \hat{p}_2)(Rd\theta)$$

$$+ \rho_2 R\Omega_2^2(\hat{\eta}Rd\theta) + T\left(\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}}\right)(Rd\theta) \qquad (4.25)$$

In the steady-state

$$P_1(Rd\theta) = P_2(Rd\theta) + \frac{T}{R}(Rd\theta) \qquad (4.26)$$

Subtracting (4.26) from (4.25), we find

$$(\hat{p}_2 - \hat{p}_1)_R + [(\rho_2 R\Omega_2^2 - \rho_1 R\Omega_1^2)\hat{n}]_R + T[\frac{1}{R} - (\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}})] = 0$$

This is exactly equation (4.23), and equation (4.24) follows for periodic solutions.

It should be emphasized that both kinematic and dynamic interfacial conditions were derived from the same set of equations for heterogeneous swirling flows without any other assumptions, and they must be satisfied when matching techniques for broken-line profiles are applied at interfaces.

Michalke and Timme (1967) examined the instability of a potential vortex enclosing a stagnant fluid core for a homogeneous density system. In matching the interfacial condition at the cylindrical vortex sheet, they only considered equating the perturbation pressures in spite of the angular velocity jump governed by equation (4.18). Uberoi et al. (1972) pointed out the error by taking into account the first order perturbation of equilibrium pressure

across the deformed surface. Their argument is essentially the following:

Equations (2.1) and (2.4) may be written as

$$P_j = \int_{R_1}^{r} \rho_j r\Omega_j^2 dr + \hat{p}_j \tag{4.27a}$$

or

$$\frac{dP_j}{dr} = \rho_j r\Omega_j^2 + \frac{d\hat{p}_j}{dr} \qquad j = 1, 2 \tag{4.27b}$$

We use the Taylor expansion with respect to the perturbation displacement $\hat{\eta}$ and neglect the second or higher order perturbation quantities. Equations (4.27) then yield

$$P_j + \hat{\eta}\frac{dP_j}{dr} = \int_{R_1}^{r} \rho_j r\Omega_j^2 dr + \hat{p}_j + (\rho_j r\Omega_j^2)\hat{\eta} \qquad j = 1, 2 \tag{4.28}$$

The total pressure together with the surface tension should be balanced at $r = R$.

$$\int_{R_1}^{R} \rho_1 r\Omega_1^2 dr + \hat{p}_1(R) + [(\rho_1 r\Omega_1^2)\hat{\eta}]_R = \int_{R_1}^{R} \rho_2 r\Omega_2^2 dr + \hat{p}_2(R) +$$

$$[(\rho_2 r\Omega_2^2)\hat{\eta}]_R + T\left(\frac{1}{R_{r\theta}} + \frac{1}{R_{rz}}\right) \tag{4.29}$$

In the steady-state

$$\int_{R_1}^{R} \rho_1 r \Omega_1^2 dr = \int_{R_1}^{R} \rho_2 r \Omega_2^2 dr + T(\frac{1}{R}) \tag{4.30}$$

Subtracting (4.30) from (4.29) and assuming period solutions, we obtain the same dynamic interfacial condition (4.18).

For piecewise continuous profiles, one can join up the solution in the proper region by use of conditions (4.12) and (4.18) at interfaces. This method, first used successfully by Kelvin, has led to explicit eigenvalue relations for many problems, which serve as an approximation to smoothly varying profiles for small wave numbers.

CHAPTER 5

SOME ANALYTICAL SOLUTIONS

As mentioned in Chapter 4, the difficulty in obtaining exact solutions to equation (2.11) arises from the singularity existing in the equation. The location of possible singularities and solutions in terms of known functions do not seem to exist. Series solutions by the Frobenius method are only obtainable when $\{W + sr\Omega - \omega\}$ and the swirling velocity gradients $\{DW + srD\Omega\}$ do not vanish simultaneously anywhere within the flow domain. This difficulty can only be overcome either by getting rid of the singularity in equation (2.11) (i.e., by considering a vortex with constant density and zero axial flow gradient) or by setting N equal to a constant, which requires constant swirling velocity profiles. All analytical solutions ever found have fallen into one or the other of these two categories: the former was first examined by Ponstein (1959) for a potential vortex superposing a liquid jet surrounded by air, and subsequently by Michalke and Timme (1967) for a vortex enclosing a stagnant core for constant density; the latter was studied by Alterman (1961) for the capillary instability of a liquid jet streaming within a surrounding

fluid, and by Reshotko and Monnin (1965) for a two-fluid wheel flow in the absence of surface tension. Solutions not falling into these two categories have not been found.

We now present examples of exact solutions of certain swirling flows with discontinuous velocity and density profiles subject to arbitrary infinitesimal perturbations.

5.1 Solutions in Terms of Step Approximations to Continuous Profiles

Consider one ring of fluid with constant density ρ_j, constant angular velocity Ω_j and constant axial velocity W_j in region j. Equations (2.9) and (2.10) for the profiles under consideration can be simplified to

$$i\rho_j\Omega_j[\frac{2m}{r} - n_jD^*]u = (k^2 + \frac{m^2}{r^2})p \tag{5.1}$$

$$i\rho_j\Omega_j(4 - n_j^2)u = (n_jDp + \frac{2m}{r}p) \tag{5.2}$$

Eliminating u from the above two equations, one obtains a differential equation for the fluctuation pressure

$$D^2p + \frac{1}{r}Dp - [k^2q_j^2 + \frac{m^2}{r^2}]p = 0 \tag{5.3}$$

where

$$q_j = \sqrt{1 - \frac{4}{n_j^2}} \qquad n_j = \frac{N_j}{\Omega_j} = k\frac{W_j}{\Omega_j} + m - \frac{\omega}{\Omega_j}$$

Equation (5.3) is the modified Bessel equation with general solutions in terms of the linear combination of the modified Bessel functions of the first and second kinds, i.e.,

$$p = -i\rho_j\Omega_j^2(n_j^2 - 4)\{Ak^{-m}I_m(\chi_j) + Bk^m K_m(\chi_j)\} \tag{5.4}$$

here

$$\chi_j = q_j kr$$

The normalizing factors k^{-m} and k^m are here introduced in order to allow $k \to 0$ for $m \neq 0$, which gives

$$\lim_{k\to 0} p = -i\rho_j\Omega_j^2(n_j^2 - 4)\{A^* r^m + B^* r^{-m}\}$$

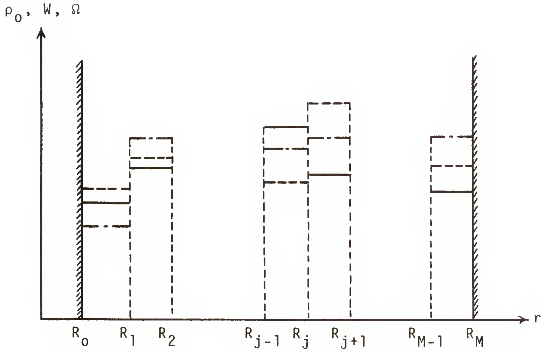
where

$$A^* = A\frac{q_j^m}{2^m m!} \qquad B^* = \frac{2^m q_j^{-m}}{(-m)!}$$

Substituting (5.4) into (5.2) leads to a solution for perturbation velocity of the form

$$u = kq_jN_j\{Ak^{-m}[\frac{\chi_j I_m'(\chi_j)}{I_m(\chi_j)} + \frac{2m}{n_j}]\frac{I_m(\chi_j)}{\chi_j}$$
$$+ Bk^m[\frac{\chi_j K_m'(\chi_j)}{K_m(\chi_j)} + \frac{2m}{n_j}]\frac{K_m(\chi_j)}{\chi_j}\} \tag{5.5}$$

To approximate continuous velocity and density profiles, consider a multiple layer system with M rings of fluid and M-1 interfaces inside the annulus between two cylindrical walls of radius R_0 and R_M, the velocity and density distributions being step functions as shown in the following sketch:



At the interface $r = R_j$, two rings of fluid j and j + 1 are connected. The displacement condition (4.12) with solutions (5.5) in region j and j + 1 gives

$$A_j k^{-m}\left[\frac{q_j\kappa_j I'_m(q_j\kappa_j)}{I_m(q_j\kappa_j)} + \frac{2m}{n_j}\right] I_m(q_j\kappa_j) + B_j k^m\left[\frac{q_j\kappa_j K'_m(q_j\kappa_j)}{K_m(q_j\kappa_j)} + \frac{2m}{n_j}\right] K_m(q_j\kappa_j)$$

$$-A_{j+1} k^{-m}\left[\frac{q_{j+1}\kappa_j I'_m(q_{j+1}\kappa_j)}{I_m(q_{j+1}\kappa_j)} + \frac{2m}{n_{j+1}}\right] I_m(q_{j+1}\kappa_j) - B_{j+1} k^m\left[\frac{q_{j+1}\kappa_j K'_m(q_{j+1}\kappa_j)}{K_m(q_{j+1}\kappa_j)}\right.$$

$$\left. + \frac{2m}{n_{j+1}}\right] K_m(q_{j+1}\kappa_j) = 0 \qquad (5.6)$$

while the pressure condition (including surface tension T_j acting at the interface) yields

$$A_j k^{-m}\{(n_j^2 - 4) + \left[\frac{2m}{n_j} + \frac{q_j\kappa_j I'_m(q_j\kappa_j)}{I_m(q_j\kappa_j)}\right][1 + \Gamma_j(1 - m^2 - \kappa_j^2)]\} I_m(q_j\kappa_j)$$

$$+B_j k^m\{(n_j^2 - 4) + \left[\frac{2m}{n_j} + \frac{q_j\kappa_j K'_m(q_j\kappa_j)}{K_m(q_j\kappa_j)}\right][1 + \Gamma_j(1 - m^2 - \kappa_j^2)]\} K_m(q_j\kappa_j)$$

$$-\alpha_{j+1,j} A_{j+1} k^{-m}\{(n_{j+1}^2 - 4) + \left[\frac{2m}{n_{j+1}} + \frac{q_{j+1}\kappa_j I'_m(q_{j+1}\kappa_j)}{I_m(q_{j+1}\kappa_j)}\right]\} I_m(q_{j+1}\kappa_j)$$

$$-\alpha_{j+1,j} B_{j+1} k^m\{(n_{j+1}^2 - 4) + \left[\frac{2m}{n_{j+1}} + \frac{q_{j+1}\kappa_j K'_m(q_{j+1}\kappa_j)}{K_m(q_{j+1}\kappa_j)}\right]\} K_m(q_{j+1}\kappa_j) = 0$$

$$(5.7)$$

where

$$\kappa_j = kR_j \qquad \Gamma_j = \frac{T_j}{\rho_j \Omega_j^2 R_j^3}$$

$$\alpha_{j+1,j} = \left(\frac{\rho_{j+1}}{\rho_j}\right)\left(\frac{\Omega_{j+1}}{\Omega_j}\right)^2$$

Equations (5.6) and (5.7) represent 2M linear homogeneous equations in the variables of interfacial displacements and pressures. Be reminded that the fluctuation velocity should vanish at the walls; hence

$$A_1 k^{-m}\left\{\frac{2m}{n_1} + \frac{q_1\kappa_o I_m'(q_1\kappa_o)}{I_m(q_1\kappa_o)}\right\} I_m(q_1\kappa_o) + B_1 k^m\left\{\frac{2m}{n_1} + \frac{q_1\kappa_o K_m'(q_1\kappa_o)}{K_m(q_1\kappa_o)}\right\} K_m(q_1\kappa_o) = 0 \tag{5.8}$$

$$A_M k^{-m}\left\{\frac{2m}{n_M} + \frac{q_M\kappa_M I_m'(q_M\kappa_M)}{I_m(q_M\kappa_M)}\right\} I_m(q_M\kappa_M) + B_M k^m\left\{\frac{2m}{n_M} + \frac{q_M\kappa_M K_m'(q_M\kappa_M)}{K_m(q_M\kappa_M)}\right\} K_m(q_M\kappa_M) = 0 \tag{5.9}$$

For nontrivial solutions, the determinant of the coefficients of (5.6) to (5.9) must be zero. This yields the dispersion relation

$$\begin{vmatrix}
I_{1,0} & K_{1,0} & & & & & & & & \\
I_{1,1} & K_{1,1} & -I_{2,1} & -K_{2,1} & & & & & & \\
I^*_{1,1} & K^*_{1,1} & -I^*_{2,1} & -K^*_{2,1} & & & & & & \\
 & & .. & .. & .. & .. & & & & \\
 & & .. & .. & .. & .. & & & & \\
 & & & & I_{j,j} & K_{j,j} & -I_{j+1,j} & -K_{j+1,j} & & \\
 & & & & I^*_{j,j} & K^*_{j,j} & -I^*_{j+1,j} & -K^*_{j+1,j} & & \\
 & & & & & & .. & .. & .. & .. \\
 & & & & & & .. & .. & .. & .. \\
 & & & & & & & & I_{M,M} & K_{M,M}
\end{vmatrix} = 0 \tag{5.10}$$

where

$$I_{j,j} = [\frac{q_j\kappa_j I'_m(q_j\kappa_j)}{I_m(q_j\kappa_j)} + \frac{2m}{n_j}] I_m(q_j\kappa_j)$$

$$K_{j,j} = [\frac{q_j\kappa_j K'_m(q_j\kappa_j)}{K_m(q_j\kappa_j)} + \frac{2m}{n_j}] K_m(q_j\kappa_j)$$

$$I_{j+1,j} = [\frac{q_{j+1}\kappa_j I'_m(q_{j+1}\kappa_j)}{I_m(q_{j+1}\kappa_j)} + \frac{2m}{n_{j+1}}] I_m(q_{j+1}\kappa_j)$$

$$K_{j+1,j} = [\frac{q_{j+1}\kappa_j K'_m(q_{j+1}\kappa_j)}{K_m(q_{j+1}\kappa_j)} + \frac{2m}{n_{j+1}}] K_m(q_{j+1}\kappa_j)$$

$$I^*_{j,j} = \{(n_j^2 - 4) + [\frac{2m}{n_j} + \frac{q_j\kappa_j I'_m(q_j\kappa_j)}{I_m(q_j\kappa_j)}][1 + \Gamma_j(1 - m^2 - \kappa_j^2)]\} I_m(q_j\kappa_j)$$

$$K^*_{j,j} = \{(n_j^2 - 4) + [\frac{2m}{n_j} + \frac{q_j\kappa_j K'_m(q_j\kappa_j)}{K_m(q_j\kappa_j)}][1 + \Gamma_j(1 - m^2 - \kappa_j^2)]\} K_m(q_j\kappa_j)$$

$$I^*_{j+1,j} = \alpha_{j+1,j}\{(n_{j+1}^2 - 4) + [\frac{2m}{n_{j+1}} + \frac{q_{j+1}\kappa_j I'_m(q_{j+1}\kappa_j)}{I_m(q_{j+1}\kappa_j)}]\} I_m(q_{j+1}\kappa_j)$$

$$K^*_{j+1,j} = \alpha_{j+1,j}\{(n_{j+1}^2 - 4) + [\frac{2m}{n_{j+1}} + \frac{q_{j+1}\kappa_j K'_m(q_{j+1}\kappa_j)}{K_m(q_{j+1}\kappa_j)}]\} K_m(q_{j+1}\kappa_j)$$

(5.11)

The generalized equations which describe determinant (5.10) are

$$A_j k^{-m} I_{j,j} + B_j k^m K_{j,j} - A_{j+1} k^{-m} I_{j+1,j} - B_{j+1} k^m K_{j+1,j} = 0 \quad (5.12)$$

$$A_j k^{-m} I^*_{j,j} + B_j k^m K^*_{j,j} - A_{j+1} k^{-m} I^*_{j+1,j} - B_{j+1} k^m K^*_{j+1,j} = 0 \quad (5.13)$$

$j = 1,2, M-1.$

with boundary conditions

$$A_1 k^{-m} I_{1,0} + B_1 k^m K_{1,0} = 0$$

$$A_M k^{-m} I_{M,M} + B_M k^m K_{M,M} = 0 \quad (5.14)$$

Determinant (5.10) is of order $2(M + 1)$ with the non-zero elements appearing in a systematic form around the diagonal. Equations (5.12), (5.13) and (5.14) give the general solution for discrete strata for any number of rings of fluid. Continuous stratifications can be approximated by letting the number of rings approach infinity ($M \to \infty$), with the thickness of each ring of fluid approaching zero. However, one should keep in mind that the velocity jump at interfaces for step-wise approximations of continuous functions will yield the Kelvin-Helmholtz instabilities for large wave numbers, while the corresponding continuous distributions are quite stable for all modes. This difference arises from the strong shearing rates existing in the sharp shear layer created at the vortex sheet. Instabilities produced by the Kelvin-Helmholtz effects due to the relative motion at interfaces are too strong to be stabilized for short wavelengths. Condition (3.19) would be violated at certain points for the discrete strata while it is actually satisfied for the corresponding continuous profiles. This phenomenon is of significant importance especially when the wave numbers are sufficiently large. The approximation of continuous distributions by many rings of fluid offers a convenient approach to eigen solutions for arbitrary continuous profiles of which exact solutions are not easily obtained. With the use of electronic computers,

reasonably good approximations for eigenvalue $\omega = \omega_r + i\omega_i$ are expected when the thickness of each layer is much less than wavelengths of the distributions under consideration.

The order of secular determinant (5.10) is reducible somewhat by a useful theorem (Pipes 1970). Considering $K_{1,0}$ and $I_{M,M}$ as the two pivotal elements for the first and second reductions of the determinant, one obtains

$$\begin{vmatrix}
I_{1,1} - H_1 K_{1,1} & -I_{2,1} & -K_{2,1} & & & & & \\
I^*_{1,1} - H_1 K^*_{1,1} & -I^*_{2,1} & -K^*_{2,1} & & & & & \\
& & \ddots & & & & & \\
& & I_{j,j} & K_{j,j} & -I_{j+1,j} & -K_{j+1,j} & & \\
& & I^*_{j,j} & K^*_{j,j} & -I^*_{j+1,j} & -K^*_{j+1,j} & & \\
& & & & \ddots & & & \\
& & & & & I_{M-1,M-1} & K_{M-1,M-1} & K_{M,M-1} - H_M I_{M,M-1} \\
& & & & & I^*_{M-1,M-1} & K^*_{M-1,M-1} & K^*_{M,M-1} - H_M I^*_{M,M-1}
\end{vmatrix} \tag{5.15}$$

$$H_1 = \frac{I_{1,0}}{K_{1,0}} \qquad H_M = \frac{K_{M,M}}{I_{M,M}} \tag{5.16}$$

If the inner cylinder vanishes $K_{1,0}$ will diverge to infinity while $I_{1,0}$ remains finite, with the result that H_1 will approach zero. For $r \to \infty$, H_M will vanish due to the divergence of $I_{M,M}$ at infinity. The order of the determinant has now been reduced to 2M, where M represents the number of rings of fluid subdivided between two solid cylinders. For some finite numbers of M, one can obtain the secular relation by following the same track to keep on reducing the order of the determinant.

With M = 2 for two region flows with an interface at r = R, the secular relation reduces to a two by two determinant

$$\begin{vmatrix} I_{1,1} - H_1 K_{1,1} & K_{2,1} - H_2 I_{2,1} \\ I^*_{1,1} - H_1 K^*_{1,1} & K^*_{2,1} \quad H_2 I^*_{2,1} \end{vmatrix} = 0 \tag{5.17}$$

Expand the determinant (5.17) by (5.11). The secular relation, after some algebraic procedures, can be written as

$$\frac{(n_1^2 - 4)[I_m(q_1\kappa) - H_1 K_m(q_1\kappa)]}{[\frac{2m}{n_1} + \frac{q_1\kappa I'_m(q_1\kappa)}{I_m(q_1\kappa)}]I_m(q_1\kappa) - H_1[\frac{2m}{n_1} + \frac{q_1\kappa K'_m(q_1\kappa)}{K_m(q_1\kappa)}]K_m(q_1\kappa)} -$$

$$\alpha\beta^2 \frac{(n_2^2 - 4)[K_m(q_2\kappa) - H_2 I_m(q_2\kappa)]}{[\frac{2m}{n_2} + \frac{q_2\kappa K'_m(q_2\kappa)}{K_m(q_2\kappa)}]K_m(q_2\kappa) - H_2[\frac{2m}{n_2} + \frac{q_2\kappa I'_m(q_2\kappa)}{I_m(q_2\kappa)}]I_m(q_2\kappa)}$$

$$= \alpha\beta^2 - 1 - \Gamma_1(1 - m^2 - \kappa^2) \qquad (5.18)$$

where

$$H_1 = \frac{\left[\frac{2m}{n_1} + \frac{q_1\kappa_1 I'_m(q_1\kappa_1)}{I_m(q_1\kappa_1)}\right] I_m(q_1\kappa_1)}{\left[\frac{2m}{n_1} + \frac{q_1\kappa_1 K'_m(q_1\kappa_1)}{K_m(q_1\kappa_1)}\right] K_m(q_1\kappa_1)}$$

$$H_2 = \frac{\left[\frac{2m}{n_2} + \frac{q_2\kappa_2 K'_m(q_2\kappa_2)}{K_m(q_2\kappa_2)}\right] K_m(q_2\kappa_2)}{\left[\frac{2m}{n_2} + \frac{q_2\kappa_2 I'_m(q_2\kappa_2)}{I_m(q_2\kappa_2)}\right] I_m(q_2\kappa_2)}$$

$$\alpha = \frac{\rho_2}{\rho_1} \qquad \beta = \frac{\Omega_2}{\Omega_1}$$

$$\kappa = kR \qquad \Gamma_1 = \frac{T}{\rho_1 \Omega_1^2 R^3}$$

$$q_j = \sqrt{1 - \frac{4}{n_j^2}} \quad n_j = k\frac{W_j}{\Omega_j} + m - \frac{\omega}{\Omega_j} \qquad j = 1,2$$

Here we consider R_1 as the inner boundary instead of R_o, and R represents the position of the interface.

Since the complex eigen amplification factor falls inside the arguments of the Bessel functions, general features of the system are difficult to see without the use

of a computer. However, some important results can be obtained by using an analytical expansion for sufficiently large or small axial wave numbers. In those cases, the Bessel functions reduce to an exponential form or a power series. Stability boundaries and growth rates for several special cases are presented in the following sections.

5.1.1 Axisymmetric Disturbances (m = 0)

For m = 0 equation (5.18) is simplified into

$$\frac{(n_1^2 - 4)\left[1 - \frac{I_0'(q_1\kappa_1)}{K_0'(q_1\kappa_1)}\frac{K_0(q_1\kappa)}{I_0(q_1\kappa)}\right]}{q_1\left[\frac{I_0'(q_1\kappa)}{I_0(q_1\kappa)} - \frac{I_0'(q_1\kappa_1)}{K_0'(q_1\kappa_1)}\frac{K_0'(q_1\kappa)}{I_0(q_1\kappa)}\right]} - \alpha\beta^2 \frac{(n_2^2 - 4)\left[1 - \frac{K_0'(q_2\kappa_2)}{I_0'(q_2\kappa_2)}\frac{I_0(q_2\kappa)}{K_0(q_2\kappa)}\right]}{q_2\left[\frac{K_0'(q_2\kappa)}{K_0(q_2\kappa)} - \frac{K_0'(q_2\kappa_2)}{I_0'(q_2\kappa_2)}\frac{I_0'(q_2\kappa)}{K_0(q_2\kappa)}\right]}$$

$$= \kappa[\alpha\beta^2 - 1 - \Gamma_1(1 - \kappa^2)] \qquad (5.19)$$

Since

$$I_0'(z) = I_1(z) \qquad K_0'(z) = -K_1(z)$$

Equation (5.19) can further be reduced to the form

$$q_1F_1N_1^2 + \alpha q_2F_2N_2^2 = \kappa\Omega_1^2[\alpha\beta^2 - 1 - \Gamma_1(1 - \kappa^2)] \qquad (5.20)$$

with

$$F_1 = \frac{I_o(q_1\kappa)}{I_1(q_1\kappa)} \frac{1 + \frac{I_1(q_1\kappa_1)}{K_1(q_1\kappa_1)} \frac{K_o(q_1\kappa)}{I_o(q_1\kappa)}}{1 - \frac{I_1(q_1\kappa_1)}{K_1(q_1\kappa_1)} \frac{K_1(q_1\kappa)}{I_1(q_1\kappa)}}$$

$$F_2 = \frac{K_o(q_2\kappa)}{K_1(q_2\kappa)} \frac{1 + \frac{I_o(q_2\kappa)}{K_o(q_2\kappa)} \frac{K_1(q_2\kappa_2)}{I_1(q_2\kappa_2)}}{1 - \frac{I_1(q_2\kappa)}{K_1(q_2\kappa)} \frac{K_1(q_2\kappa_2)}{I_1(q_2\kappa_2)}}$$

The influence of axial velocities on the stability characteristics for axisymmetric perturbations will be discussed as follows:

(a) <u>Constant axial velocities</u>

If we look for solutions with no axial flow gradient, i.e., $W_1 = W_2$, equation (5.20) takes a simple form

$$(c - W_1)^2 = \frac{R\Omega_1^2[\alpha\beta^2 - 1 - \Gamma_1(1 - \kappa^2)]}{k(q_1F_1 + \alpha q_2F_2)} \tag{5.21}$$

$c = \frac{\omega}{k}$

Since the real parts of the modified Bessel functions of the first and second kinds are monotonically increasing or decreasing functions respectively, it follows that the real parts of both F_1 and F_2 are positive definite for $\kappa_1 \leq \kappa \leq \kappa_2$. Therefore, the flow is stable whenever the

term inside the square bracket is positive, i.e.,

$$\alpha\beta^2 - 1 - \Gamma_1(1 - \kappa^2) \geq 0 \tag{5.22}$$

This result was first obtained by Alterman (1961) with neither the inner nor the outer boundary. It is not surprising that the solid walls have no effects on the stability boundary. In fact, the flow we are now considering falls into the category of centrifugal instability, in which the stability characteristics do not depend on either the walls of the cylinders or the axial wave numbers, when surface tension is neglected. This is analogous to the Rayleigh-Taylor instability in a gravitational force field; a statically unstable profile is going to initiate instabilities despite the presence of boundaries or the effects of varying wave numbers. If we look back to the derivation of the Rayleigh-Synge criterion by discussing the characteristics of the partial differential equations in the beginning of Chapter 3, we realize that the solid boundaries nowhere affected the criterion explicitly and no assumptions were made on the form of the perturbation stream function Ψ either (i.e., independent of the axial wave numbers).

Instead of solving the governing stability equation, the stability boundary for axisymmetric disturbances in the absence of surface tension and axial flow gradients

can be alternatively located through a simple integration technique. Since both flows are stable in their individual regions 1 and 2, the instabilities governed by the Rayleigh-Synge criterion can only be initiated right at the interface. In applying the integration technique with respect to this criterion across the interface, one finds the stability condition is

$$\lim_{e \to 0} \int_{R-\varepsilon}^{R+\varepsilon} D[\rho_o (r^2 \Omega)^2] = R^4 (\rho_2 \Omega_2^2 - \rho_1 \Omega_1^2) \geq 0$$

i.e.,

$$\alpha \beta^2 - 1 \geq 0 \tag{5.23}$$

Both (5.22) and (5.23) are exactly equivalent except for the surface tension term. This method can be used to locate stability boundaries for constant axial velocity subject to axisymmetric perturbations for any two region flows provided the flow in each individual region is stable. Stability boundaries are shown in Figure 5.1. Surface tension stabilizes or destabilizes the flow depending on whether κ is greater or less than one.

(b) Effects of axial velocity gradients

As previously mentioned in Chapter 3, the shear effect of axial velocity can cause instabilities of the

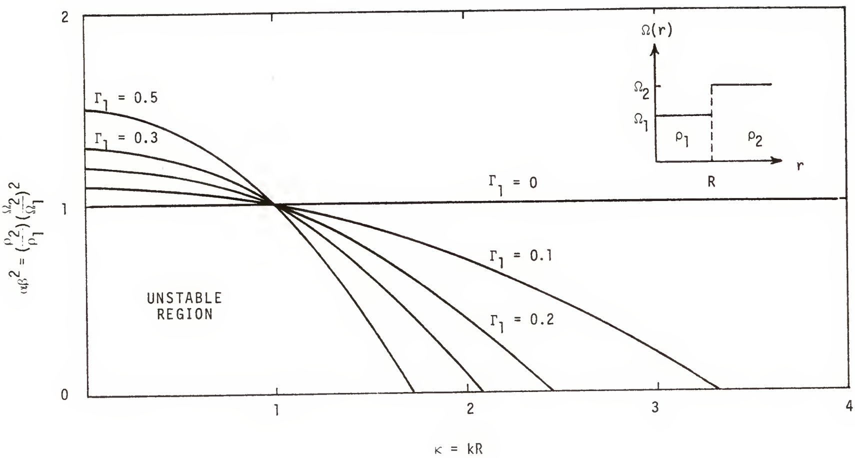


Figure 5.1 Effect of surface tension on the stability boundary for axisymmetric perturbations without axial velocity gradients.

flow even when the Rayleigh-Synge condition is satisfied. A special case from (5.19) will serve as a demonstration to this phenomenon.

For an inner boundary at $r = 0$ and outer boundary at $r = \infty$, equation (5.19) without surface tension takes the form

$$\frac{n_1^2 - 4}{\dfrac{q_1 I_0'(q_1\kappa)}{I_0(q_1\kappa)}} - \alpha\beta^2 \frac{n_2^2 - 4}{\dfrac{q_2 K_0'(q_2\kappa)}{K_0(q_2\kappa)}} = \kappa(\alpha\beta^2 - 1) \qquad (5.24)$$

Consider the neutrally stable profile (in the centrifugal sense) $\alpha\beta^2 = 1$ and expand the modified Bessel functions in their asymptotic forms valid for sufficiently large k. Equation (5.24) then yields

$$n_1\sqrt{n_1^2 - 4} + \alpha n_2\sqrt{n_2^2 - 4} = 0 \qquad (5.25)$$

Squaring (5.25) and rearranging it, one finds

$$[(W_1 - c)^2 - \alpha(W_2 - c)^2][(W_1 - c)^2 + \alpha(W_2 - c)^2 - 4(\frac{\Omega_1}{k})^2] = 0 \qquad (5.26)$$

For unstable solutions, it requires

$$[(W_1 - c)^2 + \alpha(W_2 - c)^2 - 4(\frac{\Omega_1}{k})^2] = 0$$

or

$$c = \frac{k(W_1 + \alpha W_2) \pm \Omega_1\sqrt{4(1 + \alpha) - \alpha\gamma_1^2}}{k(1 + \alpha)} \tag{5.27a}$$

where

$$\gamma_1 = \frac{k(W_2 - W_1)}{\Omega_1}$$

Unstable modes are expected whenever

$$\alpha\gamma_1^2 > 4(1 + \alpha) \tag{5.27b}$$

This example clearly demonstrates that the Rayleigh-Synge criterion fails to be valid if the axial velocity gradient is taken into consideration (Chandrasekhar 1960b), and that the shear instability for centrifugally stable profiles is governed by condition (3.19).

Figures 5.2a,b show the stability boundary, the growth rate and the wave speed. It is interesting to note that the complex wave velocity falls inside a semi-circle as predicted in category (b), Section (3.3.1) of Chapter 3. Furthermore, all unstable solutions for short waves lie exactly on a smaller semi-circle governed by the equation

$$\{c_r - (W \pm |c_i|_{max})\}^2 + c_i^2 = |c_i|_{max}^2 \tag{5.28}$$

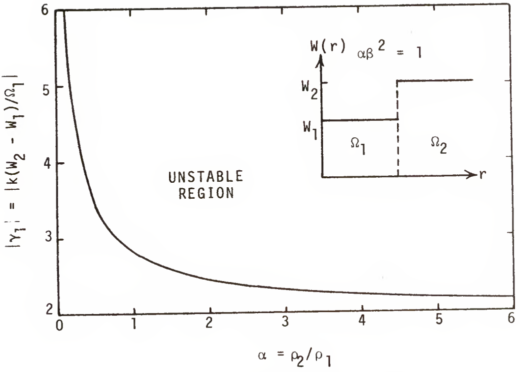


Figure 5.2a Stability boundary for axisymmetric disturbances due to the axial velocity difference.

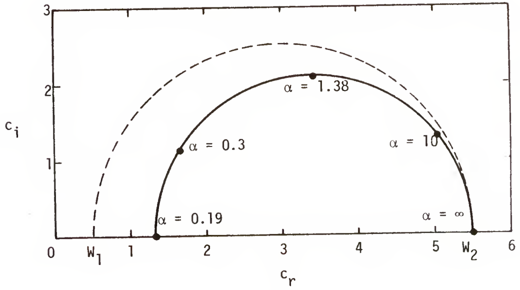


Figure 5.2b Density effect on the complex wave speed for the centrifugally stable profile $\alpha\beta^2 = 1$ ($\Omega_1/k = 1$).

where

$$|c_i|_{max} = \frac{\gamma_1^2 - 4}{2k}\left|\frac{\Omega_1}{\gamma_1}\right|$$

obtained by differentiating the imaginary part of (5.27a) with respect to α for unstable solutions. Here the plus or minus sign in (5.28) depends on whether the value $\{W_1 - W_2\}$ is positive or negative respectively. Either large or small density ratio has a stabilizing effect. For $\alpha \leq 0.19$ or $\alpha = \infty$, all the shear instabilities will be completely stabilized by this density effect and the flow will be stable. A liquid jet directed into a rotating air environment or a rotating air jet blowing into a heavy density medium is stable under axisymmetric disturbances provided surface tension is neglected. Growth rates are expected to increase as the densities of two fluids approach each other, and to reach a maximum at $\alpha = (\gamma_1^2 + 4)/(\gamma_1^2 - 4) = 1.38$.

5.1.2 Azimuthally periodic perturbations (k = 0)

From the general discussions of the criteria for rotating flows in Chapter 3, it is suggested that the two-dimensional perturbations for k = 0 grow more rapidly than the corresponding three-dimensional ones. Azimuthally periodic disturbances in rotating flows can

therefore not be neglected in stability investigations.

For small axial wave numbers ($k << 1$), the modified Bessel function in (5.18) can be approximated by their algebraic forms for small arguments. The secular relation for the two-region flow is then rearranged into the following form after some algebraic procedures.

$$n_1(f_1 n_1 - 2) + \alpha\beta^2 n_2(f_2 n_2 + 2) = m[\alpha\beta^2 - 1 - \Gamma_1(1 - m^2 - \kappa^2)] \quad (5.29)$$

where

$$f_j = \frac{1 + \varepsilon_j^{2m}}{1 - \varepsilon_j^{2m}} \geq 1 \qquad j = 1,2$$

$$\varepsilon_1 = \frac{R_1}{R}$$

$$\varepsilon_2 = \frac{R}{R_2}$$

Equation (5.29) is a quadratic form for ω with the solution

$$\frac{\omega}{\Omega_1} = \frac{[f_1(k\frac{W_1}{\Omega_1} + m) - 1] + \alpha[f_2(k\frac{W_2}{\Omega_1} + m\beta) + \beta] \pm \sqrt{\Delta}}{f_1 + \alpha f_2} \quad (5.30)$$

where

$$\Delta = (f_1 + \alpha f_2)[(\frac{1}{f_1} - m) + \alpha\beta^2(\frac{1}{f_2} + m) - m\Gamma_1(1 - m^2 - \kappa^2)]$$

$$- \alpha f_1 f_2[\gamma + (\frac{1}{f_1} - m) + \beta(\frac{1}{f_2} + m)]^2$$

The stability boundary is described explicitly by the characteristic equation

$$\Delta = 0 \tag{5.31}$$

For unbounded flow ($R_1 \to 0$ and $R_2 \to \infty$) both boundary factors $f_1 = f_2 = 1$, and (5.30) reduces to

$$\frac{\omega}{\Omega_1} = \frac{(k\frac{W_1}{\Omega_1} + m - 1) + \alpha(k\frac{W_2}{\Omega_1} + m\beta + \beta) \pm \sqrt{\Delta}}{1 + \alpha} \tag{5.32a}$$

with the stability boundary at

$$\Delta = (1 + \alpha)[(1 - m) + \alpha\beta^2(1 + m) - m\Gamma_1(1 - m^2 - \kappa^2)]$$

$$- \alpha[\gamma + (1 - m) + \beta(1 + m)]^2 = 0 \tag{5.32b}$$

We make several observations from (5.32b) and summarize them as follows:

(a) <u>Effects of azimuthal wave numbers</u>

For k = 0 and in the absence of surface tension, equation (5.32b) can be solved for

$$\beta = \frac{\alpha(m^2 - 1) \pm \sqrt{m\alpha(m^2 - 1)(\alpha^2 - 1)}}{\alpha(m + 1)(m - \alpha)} \tag{5.33}$$

Stability boundaries for different values of m are plotted out and compared to that of axisymmetric disturbances in (5.23). These axisymmetric disturbances will be stable whenever $\alpha\beta^2 \geq 1$ and unstable when $\alpha\beta^2 < 1$. At the same time, the stability boundaries corresponding to azimuthal modes are observed to lie partially in the region that is stable to axisymmetric modes. This means that the present flow can be quite stable to axisymmetric disturbances in part of the α-β plane and yet unstable to non-axisymmetric modes. Such a centrifugally stable profile can be upset by the shear effects, once they are sufficiently strong. Indeed, as $m \to \infty$ such azimuthally periodic modes will amplify in time everywhere except along the line $\beta = 1$, $\alpha \geq 1$. These results clearly demonstrate that the Rayleigh-Synge criterion is generally neither a necessary nor a sufficient condition for the stability of a heterogeneous swirling flow when allowance is made for non-axisymmetric disturbances, and show that the non-axisymmetric modes of instability, especially corresponding to $k = 0$ which amplify most strongly in this case, cannot be neglected when discussing the stability of such flows.

Figure 5.4 shows the instability patterns for different azimuthal periodic modes when $\beta = 1$. The stream function, which represent the paths of moving particles, has the form

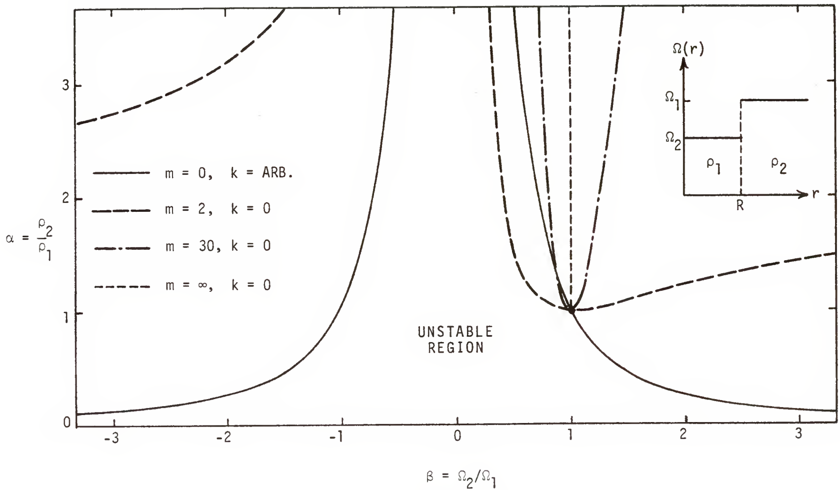


Figure 5.3 Stability boundaries for axisymmetric (m=0) and azimuthally periodic (k=0) perturbations.

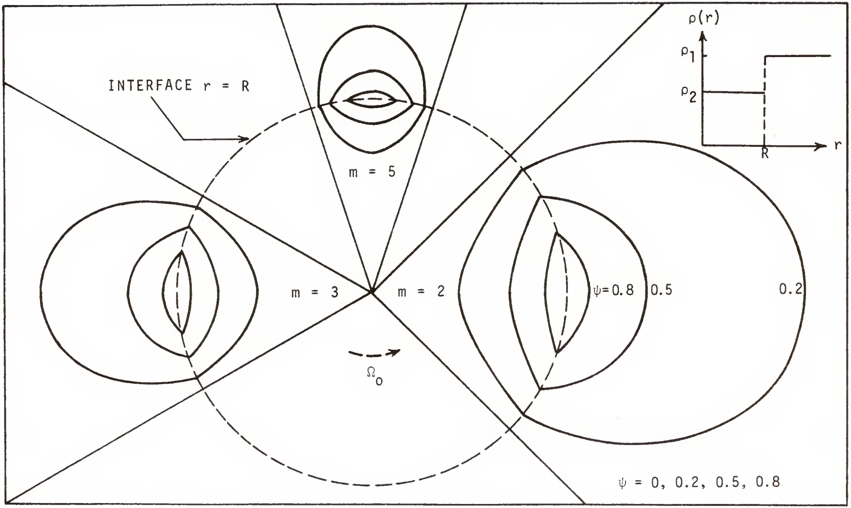


Figure 5.4 Stability patterns of rigid body rotation for azimuthally periodic modes with a density difference at the interface.

$$\psi = \begin{cases} \left(\frac{r}{R}\right)^m \cos m\theta & \text{for} \quad r \le R \\ \left(\frac{R}{r}\right)^m \cos m\theta & \text{for} \quad r \ge R \end{cases}$$

These patterns would be observed in a reference frame rotating at the constant angular velocity of the flow.

(b) Effects of surface tension

Equation (5.32b) for $k = 0$ can be rearranged as

$$\beta = \frac{\alpha(m^2 - 1) \pm \sqrt{\alpha m(\alpha + 1)(m^2 - 1)[(\alpha - 1) - (\alpha - m)(m + 1)\Gamma_1]}}{\alpha(m + 1)(m - \alpha)} \tag{5.34}$$

Figure 5.5 shows the effects of surface tension on the stability boundaries for different values of m in the $\alpha - \beta$ plane. Several observations follow immediately from the behavior of the curves. Surface tension stabilizes the flow especially when the relative velocity at the interface is small. This stabilizing effect will not be significant when stabilities are dominated by strong shearing rates existing at the interface, i.e., two rings of fluids rotating in opposite direction or one ring rotating a lot faster than another. The effect of surface tension becomes significantly important as m increases. The stable region in Figure 5.5 will expand more and more as m becomes larger and larger and eventually

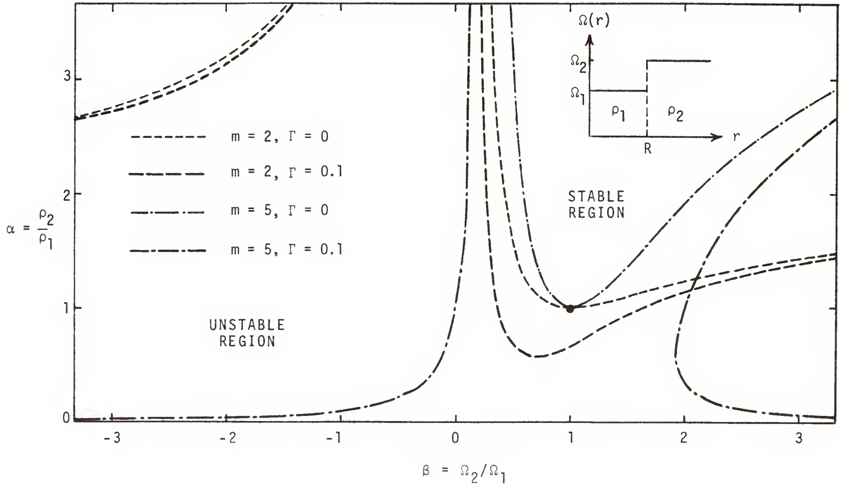


Figure 5.5 Effects of surface tension on stability boundaries for azimuthally periodic disturbances.

it will cover the whole α-β plane as $m \to \infty$. This means that the most unstable mode cannot exist under the influence of surface tension. The capillary forces existing at the interface are sufficiently strong to damp down the onset of instability corresponding to very short azimuthal wave lengths.

(c) <u>Growth rates corresponding to the complex semi-circle</u>

Consider $\alpha > 1$ and $\alpha\beta = 1$ for azimuthally periodic disturbances. Equation(5.32a) without surface tension can be simplified into

$$\frac{\hat{c}}{\Omega_1} = \frac{2}{1 + \alpha} \pm i\frac{\sqrt{m(\alpha - 1)^2 - (\alpha^2 - 1)}}{\sqrt{m\alpha}(1 + \alpha)} \qquad (5.35a)$$

with

$$\hat{c} = \frac{\omega}{m}$$

The flow will be unstable whenever

$$\alpha > \frac{m + 1}{m - 1} \qquad \text{i.e.,} \quad \beta < \frac{m - 1}{m + 1} \qquad (5.35b)$$

The solutions for complex wave speed are shown in Figure 5.6. As previously predicted by (3.33), the complex azimuthal wave velocity ($\hat{c} = \frac{\omega}{m}$) lies within a certain semi-circle of diameter equal to the range of rotating velocity. The growth rate amplifies as m increases

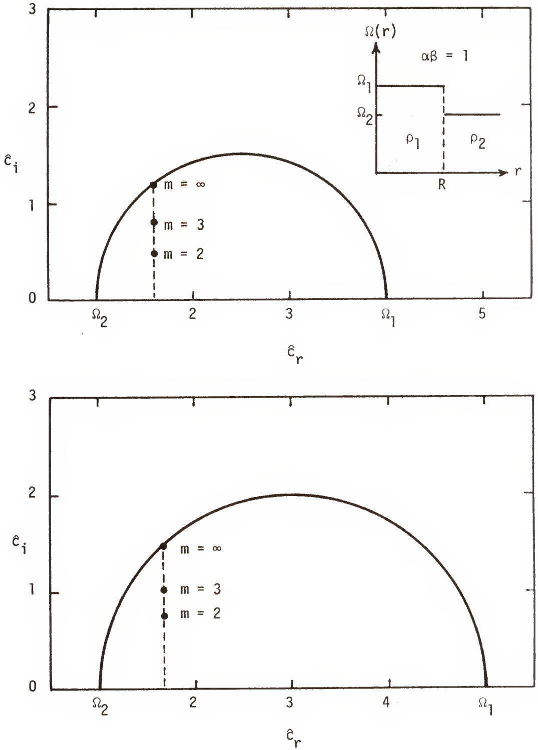


Figure 5.6 Effect of azimuthal wave numbers on the complex eigen phase velocity for azimuthally periodic disturbances.

until it touches the periphery of the semi-circle for $m \to \infty$, which represents the upper bound upon the maximum growth rate for the present flow.

(d) Effects of axial flows for small k

The presence of axial flows is irrelevant for $k = 0$. These effects are worth considering however for small axial wave numbers (for example, $k = 0.1$) if variations of axial flow are significant. Equation (5.32b) without surface tension yields

$$(1 + \alpha)[(1 - m) + \alpha\beta^2(1 + m)] - \alpha[\gamma + (1 - m) + \beta(1 + m)]^2 = 0 \tag{5.36}$$

From Figure 5.7 we will notice immediately that the stability boundaries depart from the point $\alpha = 1$, $\beta = 1$ once the axial flow difference is taken into consideration. The lower the azimuthal wave number, the more the stability boundaries depart from that point. Another phenomenon is that the positive γ_1 has a destabilizing effect when two rings of fluids are rotating in the same direction and stabilizing effects when they are rotating in opposite directions. These effects will diminish for larger azimuthal wave numbers and finally have no influence on the stability boundaries as $m \to \infty$.

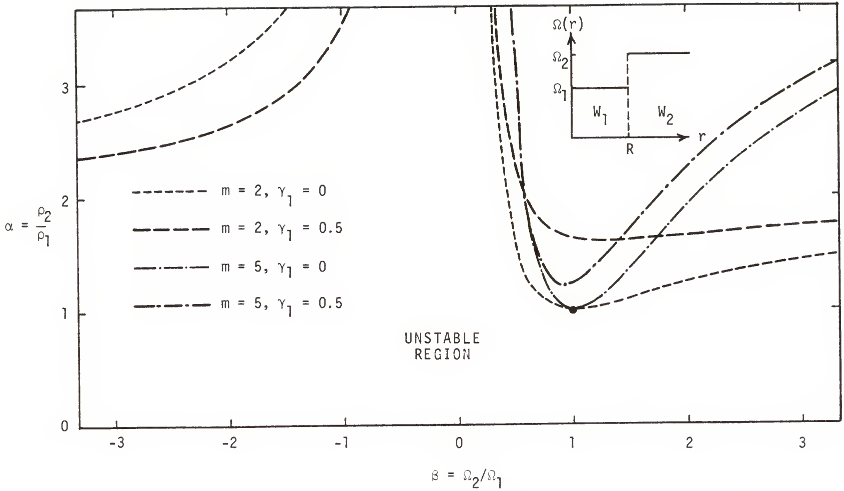


Figure 5.7 Effects of axial velocity differences on stability boundaries for small axial wave numbers.

(e) Effects of boundaries

It has been shown previously that the solid walls do not have any effect on the stability boundaries if only axisymmetric disturbances are considered, since the centrifugal instability depends only on the centrifugal pressure balance between two adjacent fluid elements. However, this argument will not hold any longer once shear effects become a dominant factor in the initiation of instabilities.

The stability boundary reduced from (5.31) in the absence of surface tension becomes

$$\beta = \frac{\alpha(mf_1 - 1)(mf_2 + 1) \pm \sqrt{\alpha m(\alpha - 1)(f_1 + \alpha f_2)(mf_1 - 1)(mf_2 + 1)}}{\alpha(mf_1 - \alpha)(mf_2 + 1)} \tag{5.37}$$

and is plotted in the α-β plane for different values of boundary ratios ε_1 and ε_2. As we notice from Figure 5.8, the solid boundaries tend to destabilize the flow for small azimuthal wave numbers as they are brought in. This destabilization is at first sight unexpected since it contradicts what would be expected in the case of continuous profiles as the walls are brought sufficiently close to the interface. However, the mechanism behind it can be understood by considering a two-fluid system between walls, with constant density and velocity in each region in a two-dimensional gravitaitonal flow field as shown:

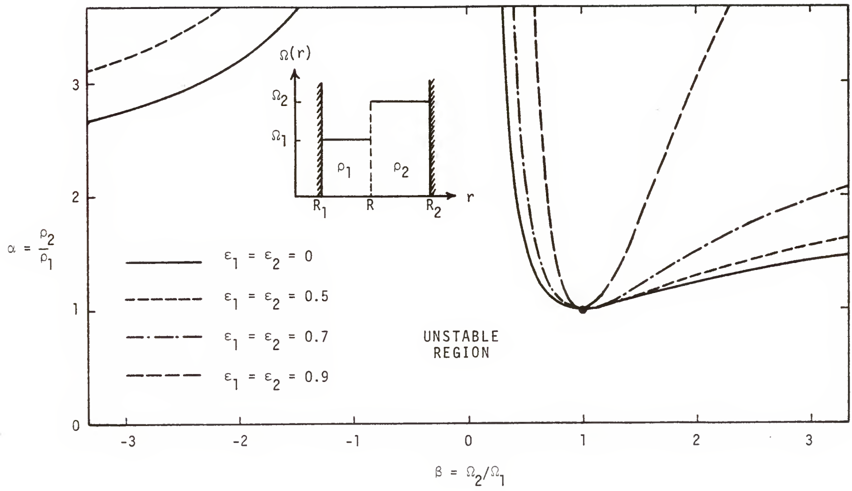
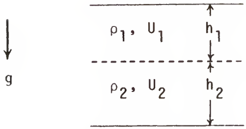


Figure 5.8 Effect of solid walls on the stability boundary for m = 2 subject to azimuthally periodic disturbances.



A well-known result for interfacial waves in the absence of surface tension for given values of wave number k, obtained by Greenhill (1887), is

$$k\rho_1(U_1 - \bar{c})^2 \cot h(kh_1) + k\rho_2(U_2 - \bar{c})^2 \cot h(kh_2) = g(\rho_2 - \rho_1) \quad (5.38)$$

Here $\bar{c}$ is complex eigen velocity to be determined. Consider long wave solutions for finite wall distances. Then (5.38) gives

$$\rho_1\frac{(U_1 - \bar{c})^2}{h_1} + \rho_2\frac{(U_2 - \bar{c})^2}{h_2} \approx g(\rho_2 - \rho_1)$$

or

$$\bar{c} = \frac{\frac{U_1}{h_1} + \alpha\frac{U_2}{h_2} \pm \sqrt{-\frac{\alpha}{h_1h_2}(U_1 - U_2)^2 + (\frac{1}{h_1} + \frac{\alpha}{h_2})(\alpha - 1)g}}{\frac{1}{h_1} + \frac{\alpha}{h_2}} \quad (5.39)$$

α is the density ratio at the interface. The destabilization of moving the walls inwards can be observed immediately by assuming $\rho_1 = \rho_2$ (i.e., $\alpha = 1$) and is similar to what we observe in a rotating case.

This boundary effect for the present rotating flow will diminish as m increases and eventually it will not influence the stability boundary for sufficiently short azimuthal waves.

5.1.3 The Non-rotating Jets

The solution of two concentric non-rotating jets with different densities in the absence of solid boundaries has been obtained by Alterman (1961), and the stability characteristics for some cases have been discussed. We would like to bring up here some of the interesting results concerning the bound on the complex eigen velocity ($c = \frac{\omega}{k}$) as derived in the discussion of the general criteria.

For $\Omega_1 = \Omega_2 = 0$, equation (5.18) can be arranged as

$$E_1(W_1 - c)^2 - \alpha E_2(W_2 - c)^2 = -\Gamma(1 - m^2 - \kappa^2) \quad (5.40)$$

where

$$E_1 = \frac{1}{\kappa}\frac{I_m(\kappa)K_m'(\kappa_1) - I_m'(\kappa_1)K_m(\kappa)}{I_m'(\kappa)K_m'(\kappa_1) - I_m'(\kappa_1)K_m'(\kappa)}$$

$$E_2 = \frac{1}{\kappa}\frac{I_m'(\kappa_2)K_m(\kappa) - I_m(\kappa)K_m'(\kappa_2)}{I_m(\kappa_2)K_m'(\kappa) - I_m'(\kappa)K_m'(\kappa_2)}$$

$$\Gamma = \frac{T}{\rho_1 k^2 R^3}$$

Solving for the complex c, one finds

$$c = \frac{E_1W_1 - \alpha E_2W_2 \pm\sqrt{\alpha E_1E_2(W_2 - W_1)^2 - (E_1 - \alpha E_2)\Gamma(1 - m^2 - \kappa^2)}}{E_1 - \alpha E_2}$$

(5.41)

The condition for stability is

$$\alpha E_1E_2(W_2 - W_1)^2 - (E_1 - \alpha E_2)\Gamma(1 - m^2 - \kappa^2) \geq 0 \qquad (5.42)$$

Since

$$I_m'(\kappa_2) \geq I_m'(\kappa) \geq I_m'(\kappa_1)$$

$$K_m'(\kappa_2) \geq K_m'(\kappa) \geq K_m'(\kappa_1) \qquad \text{for } \kappa_2 \geq \kappa \geq \kappa_1$$

it follows that

$$E_1 \geq 0 \qquad E_2 \leq 0$$

We notice immediately that (5.42) is always violated, i.e., instability occurs whenever

$$\Gamma(1 - m^2 - \kappa^2) \leq 0$$

Surface tension has stabilizing effects except for $m = 0$ with $\kappa < 1$, and the flow is always unstable in the absence of surface tension.

For unbounded flow, $E_1 = \dfrac{I_m(\kappa)}{\kappa I_m'(\kappa)}$, $E_2 = \dfrac{K_m(\kappa)}{\kappa K_m'(\kappa)}$ equation (5.40) becomes

$$\alpha(W_2 - W_1)^2 + [\frac{\kappa K_m'(\kappa)}{K_m(\kappa)} - \alpha\frac{\kappa I_m'(\kappa)}{I_m(\kappa)}]\Gamma(1 - m^2 - \kappa^2) \leq 0 \qquad (5.43)$$

This result was originally due to Alterman (1961).

For maximum growth rates, we differentiate the imaginary part of the wave speed c for unstable solutions with respect to the density ratio α in (5.41) and obtain

$$|c_i|_{max} = \frac{1}{2}[|W_1 - W_2| + \frac{\Gamma(1 - m^2 - \kappa^2)}{E_1|W_1 - W_2|}] \qquad (5.44)$$

with the corresponding real wave speed

$$c_r = \frac{1}{2}[(W_1 + W_2) + \frac{\Gamma(1 - m^2 - \kappa^2)}{E_1(W_1 - W_2)}]$$

and density ratio

$$\alpha = -\frac{E_1}{E_2}\frac{[E_1(W_1 - W_2)^2 - \Gamma(1 - m^2 - \kappa^2)]}{[E_1(W_1 - W_2)^2 + \Gamma(1 - m^2 - \kappa^2)]}$$

If we plot the unstable solutions in the complex c plane, one of the very interesting results observed is that the complex eigen values lie exactly on a certain semi-circle governed by the equation

$$\{c_r - (W_2 \pm |c_i|_{max})\}^2 + \{c_i\}^2 = |c_i|^2_{max} \qquad (5.45)$$

where the positive sign corresponds to $W_1 > W_2$ and the negative sign corresponds to $W_1 < W_2$. The locus of maximum

growth rate is found to be a straight line in the complex c plane with the equation:

$$c_i - c_r = \frac{1}{2}[|W_1 - W_2| - (W_1 + W_2)] \qquad (5.46)$$

For $W_1 = W_2$, the only instability mechanism is due to the surface tension existing at the interface, the flow will only be unstable when $m = 0$ and $\kappa < 1$. This situation was first discovered by Rayleigh (1892b) for unbounded flows.

Figure 5.9 shows the surface tension effect on the unstable solutions subject to axisymmetric disturbances for different values of κ in the complex c plane. A larger axial wave number corresponds to a smaller semi-circle. All these semi-circles will eventually shrink to a point at $c_i = 0$ and $c_r = W_2$, where the density ratio α is equal to infinity, and the flow is stable. If we look back to (5.42), we notice that the damping effects caused by surface tenison at the interface will be significant for sufficiently large axial and azimuthal wave numbers. In such cases, the curvature of the disturbed interface will be large enough to produce damping forces to stabilize the onset of instability. Therefore, the unstable solutions for sufficiently short wave lengths will not exisit due to these stabilizing effects.

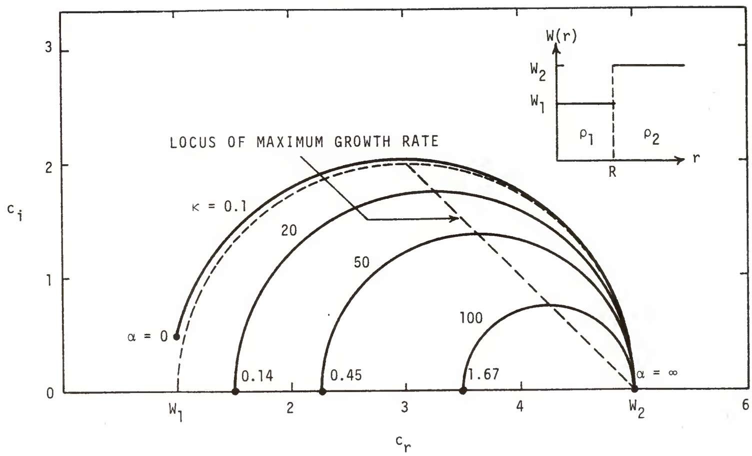


Figure 5.9 Effect of surface tension ($\Gamma = 0.1$) on the complex wave speed for axisummetric disturbances.

In the absence of the surface tension, equation (5.45) gives

$$\{c_r - \tfrac{1}{2}(W_1 + W_2)\}^2 + c_i^2 = \{\tfrac{1}{2}(W_1 - W_2)\}^2 \qquad (5.47)$$

This is also very interesting as the above equation is exactly the semi-circle bound on the complex c given by (3.29). All the solutions (in this case, they are all unstable) lie not inside, but exactly on, a semi-circle with the range of the axial velocity as diameter. A question may be raised at this point how the exactness between the predicted bound and the solution can ever be achieved even in the presence of the density and boundary effects. In fact, this can be explained by looking back to the derivation of the semi-circle bound on the complex wave velocity for non-rotating jets.

Combining equations (3.28), one obtains

$$0 \leq \int_{R_1}^{R_2} (W - a)(W - b)Qdr = \{[c_r - \tfrac{1}{2}(a + b)]^2 + c_i^2 - [\tfrac{1}{2}(a - b)]^2\} \int_{R_1}^{R_2} Qdr \qquad (5.48)$$

For the flow under consideration, the axial velocity profile is

$$W = W_1 + (W_2 - W_1)H(r - R) \qquad (5.49)$$

with the lower bound of the velocity $W_1 = a$ and the upper bound of the velocity $W_2 = b$, assuming $W_2 > W_1$. $H(r - R)$ stands for the Heaviside unit step function. From (5.48)

$$\int_{R_1}^{R_2} (W - W_1)(W - W_2) Q dr$$

$$= \lim_{\varepsilon \to o} \left\{ \int_{R_1}^{R-\varepsilon} + \int_{R-\varepsilon}^{R+\varepsilon} + \int_{R+\varepsilon}^{R_2} \right\} (W - W_1)(W - W_2) Q dr$$

$$= \lim_{\varepsilon \to o} \int_{R-\varepsilon}^{R+\varepsilon} [(W_2 - W_1) H(r - R)][(W_1 - W_2) + (W_2 - W_1) H(r - R)] Q dr$$

$$= \lim_{\varepsilon \to o} (W_2 - W_1)^2 \int_{R-\varepsilon}^{R+\varepsilon} [H(r - R) - 1]\, H(r - R)\, Q dr \to 0 \qquad (5.50)$$

In this case, the inequality (5.48) becomes an equality, and, accordingly, all the unstable solutions with the same axial velocity profile must lie exactly on the semi-circle described by (3.29) or (5.47) irrespective of either the density distributions or the boundary effects. The solutions for very small or very large density ratios lie on the two ends of the semi-circle representing the stabilizing effect of density on the growth rates.

Figure 5.10a shows the maximum growth rates ($c_i = \frac{1}{2}|W_2 - W_1|$) for different values of κ and m for unbounded flows. Solutions for axisymmetric disturbances are characteristically different from those for non-axisymmetric ones. For sufficiently large azimuthal or axial wave numbers, the curves for maximum growth rate approach $\alpha = 1$, i.e., the flow will be most unstable when the densities of the fluids approach each other at the interface. This behavior is similar to that of the Kelvin-Helmholtz instability encountered in a two-layer heterogeneous shear flow (Lamb 1932, §232). The maximum growth rate for sufficiently large wave numbers, or in the absence of the gravitational and surface tension effects, occurs when the densities in the two layers are equal.

The effects of the solid boundaries for axisymmetric disturbances are plotted in Figures 5.10b,c,d. The inner boundary travelling outwards will move the curve for maximum growth rate up in the α - κ plane while the outer boundary travelling inwards will move it down. If we bring in both boundaries sufficiently close, the curve for maximum growth rate will approach the line $\alpha = 1$. Indeed, the flow will always be stable for all azimuthal wave numbers if $k = 0$.

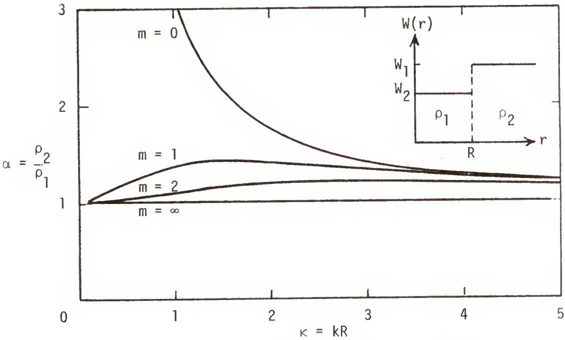


Figure 5.10a Curves of maximum growth rate for different values of wave numbers.

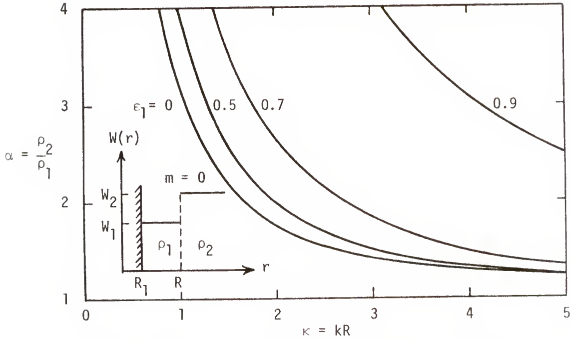


Figure 5.10b Curves of maximum growth rate for different values of inner boundary ratio ε_1.

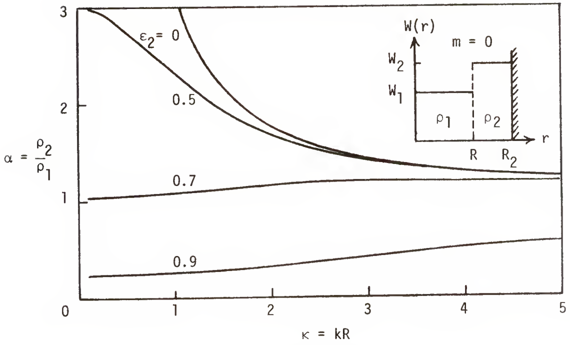


Figure 5.10c Curves of maximum growth rate for different values of outer boundary ratio ε_2.

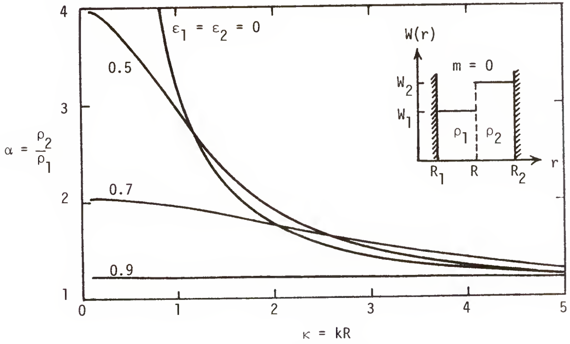


Figure 5.10d Curves of maximum growth rate for different values of boundary ratios ($\varepsilon_1 = \varepsilon_2$).

5.2 A Rotating Jet Core with Continuous Density Distribution Embedded in a Potential Vortex of Different Density

The stability of a potential vortex with axial motions was first studied analytically by Ponstein (1959) including the surface tension and viscosity of the liquid, and the inertial effects of the ambient air. He found that in some cases non-axisymmetric perturbations grew more rapidly than axisymmetric ones and instabilities took place if the wave numbers in both directions were less than some critical values. Also, the influence of the inertial effects of the ambient air on the solid jet might be neglected if the density of the air was small compared to that of the liquid and provided that the velocity field was continuous across the interface. Michalke and Timme (1967) approximated a single vortex, as it occurs in a free boundary layer, by an axisymmetrical vortex model of a homogeneous fluid subject to inviscid stability analysis. One of the important features they pointed out is that the two-dimensional azimuthally periodic perturbations are apparently more strongly amplified than the corresponding three-dimensional ones. This has been observed experimentally by Weske and Rankin (1963) in a special arrangement. However, in one of the three special types of vortex they examined, they failed to consider

the centrifugal effects arising from the discontinuity of velocity field across the interface, which introduced error into their analysis. Later, Uberoi et al. (1972) looked at a rotating jet embedded in a single vortex with different densities and obtained approximate expressions for the complex wave numbers valid for very long and very short axial wave lengths. Recently, Lessen et al. (1973) investigated a similar problem in the limiting case of homogeneous fluid and procured numerical solutions for a non-rotating and rotating jet core with no discontinuity in azimuthal velocity across the interface.

In the following example we investigate the stability of a cylindrical potential vortex surrounding a rotating jet core with continuous density variation.

The flow under consideration consists of two regions of fluid with an interface at $r = R$ and the inner and outer solid boundaries at R_1 and R_2 respectively. The velocity and density distributions in regions 1 and 2 are given as

$$\left\{\begin{array}{l} \Omega = \Omega_1 \\ W = W_1 \\ \rho_0 = \rho_1 \left(\frac{r}{R}\right)^{\sigma} \end{array}\right. \qquad \text{for } R_1 \leq r < R$$

$$\begin{cases} \Omega = \Omega_2\left(\frac{R}{r}\right)^2 \\ W = W_2 \\ \rho_o = \rho_2 \end{cases} \qquad \text{for } R < r \leq R_2$$

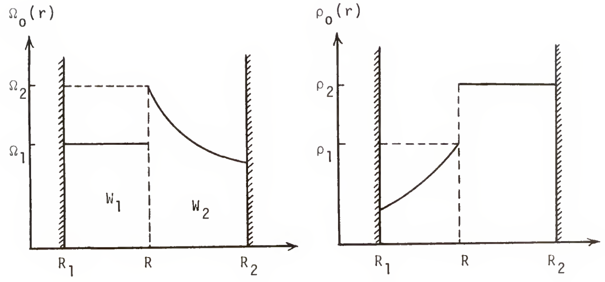


Figure 5.11 Velocity and density distributions for a rotating jet core embedded in a potential vortex.

Here ρ_1, ρ_2, Ω_1, Ω_2, W_1, W_2 and σ are real constants.

Before proceeding to the analysis of the flow, we would like first to discuss the stability characteristics concerning the velocity and density profiles in region 1.

Due to the difficulty in solving equation (2.11) for continuous velocity and density distributions, analytical

solutions have up to now been confined to broken-line type profiles in velocity and especially step function type profiles in density. The smoothly varying density distribution we are now discussing is a break from these confines.

Equations (2.9) and (2.10) for the profiles in region 1 reduce to

$$n_1 Dp_1 + \frac{2m}{r}p_1 = -i\Omega_1\rho_1\left(\frac{r}{R}\right)^{\sigma}[(\sigma + 4) - n_1^2]u_1 \tag{5.51}$$

$$\left(k^2 + \frac{m^2}{r^2}\right)p_1 = i\Omega_1\rho_1\left(\frac{r}{R}\right)^{\sigma}\left(\frac{2m}{r} - n_1 D^*\right)u_1 \tag{5.52}$$

where

$$n_1 = \frac{N_1}{\Omega_1} = k\frac{W_1}{\Omega_1} + m - \frac{\omega}{\Omega_1}$$

Eliminating u_1 from the above two equations, we obtain

$$D^2 p_1 + \frac{1 - \sigma}{r}Dp_1 - \left(\frac{\bar{\nu}^2}{r^2} + q_2^2 k^2\right)p_1 = 0 \tag{5.53}$$

where

$$q = \sqrt{1 - \frac{4 + \sigma}{n_1^2}}$$

$$\bar{\nu} = \sqrt{m^2\left(1 - \frac{\sigma}{n_1^2}\right) + \frac{2m\sigma}{n_1}}$$

Upon letting $p_1 = r^{\frac{\sigma}{2}} y$, equation (5.53) can be transformed into a modified Bessel equation

$$D^2Y + \frac{1}{r}DY - \left(\frac{\nu^2}{r^2} + q^2k^2\right)Y = 0 \tag{5.54}$$

with solutions in the form of the modified Bessel function of the first and second kinds, namely

$$Y = A^* I_\nu(qkr) + B^* K_\nu(qkr) \tag{5.55}$$

where

$$\nu = \sqrt{m^2q^2 + \left(\frac{2m}{n_1} + \frac{\sigma}{2}\right)^2} = \sqrt{m^2 + \sigma\Lambda + \left(\frac{\sigma}{2}\right)^2}$$

The solution for the perturbation pressure can now be written as

$$p_1 = -i\Omega_1^2\rho_1\left(\frac{r}{R}\right)^\sigma (n_1^2 - 4 - \sigma)\{r^{-\frac{\sigma}{2}}[Ak^{-\nu}I_\nu(qkr) + Bk^\nu K_\nu(qkr)]\} \tag{5.56}$$

where $k^{-\nu}$ and k^ν are normalizing factors in order to allow $k \to 0$ for $\nu \neq 0$.

Substituting (5.56) into (5.51), we obtain the solution for the perturbation velocity in the following form.

$$u_1 = N_1 r^{-\frac{\sigma}{2} - 1}\{Ak^{-\nu}[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{qkrI_\nu'(kqr)}{I_\nu(kqr)}]I_\nu(qkr) +$$

$$Bk^{\nu}[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{qkrK_\nu'(kqr)}{K_\nu(kqr)}]K_\nu(qkr)\} \qquad (5.57)$$

Here we consider the profile in region 1 to be within the annulus of two solid walls at $r = R_1$ and R_2 for the time being. Some of the stability conditions for this continuously varying density profile can be revealed by making use of the boundary conditions that the perturbation velocity must vanish at r equal to R_1 and R_2. This yields

$$\begin{vmatrix} [\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa_1 I_\nu'(q\kappa_1)}{I_\nu(q\kappa_1)}]I_\nu(q\kappa_1) & [\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa_1 K_\nu'(q\kappa_1)}{K_\nu(q\kappa_1)}]K_\nu(q\kappa_1) \\ [\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa_2 I_\nu'(q\kappa_2)}{I_\nu(q\kappa_2)}]I_\nu(q\kappa_2) & [\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa_2 K_\nu'(q\kappa_2)}{K_\nu(q\kappa_2)}]K_\nu(q\kappa_2) \end{vmatrix} = 0 \qquad (5.58)$$

Since the amplification factor ω is inside not only the argument, but also the order of the modified Bessel functions, analytical solutions are not possible except for special conditions. Two particular cases will be discussed and compared with the stability criteria:

(a) For axisymmetric disturbances, equation (5.58) can further be reduced to

$$\frac{I_\mu(q\kappa_1)}{I_\mu(q\kappa_2)} = \frac{K_\mu(q\kappa_1)}{K_\mu(q\kappa_2)} \tag{5.59}$$

where

$$\mu = |1 - \sigma|$$

An asymptotic expansion of the Bessel functions in (5.59) yields

$$e^{2q(\kappa_2 - \kappa_1)} = 1 = e^{2a\pi i} \qquad a = 0, 1, 2 \ldots$$

Recalling q in (5.53) and solving for ω, one finds,

$$\frac{\omega}{\Omega_1} = k\frac{W_1}{\Omega_1} + m \pm \sqrt{\frac{\sigma + 4}{1 + \frac{a^2\pi^2}{(\kappa_2 - \kappa_1)^2}}} \tag{5.60}$$

ω will be complex if $\sigma < -4$, which is exactly what the Rayleigh-Synge condition predicts. In fact, the axi-symmetric modes will be stable for all axial wave lengths whenever $\sigma \geq -4$, according to that condition.

(b) For small axial wave numbers, the modified Bessel functions take their algebraic forms and (5.58) is re-ducible to a very simple form

$$\left(\frac{R_2}{R_1}\right)^{2\nu} = 1 = e^{2a\pi i} \qquad a = 0, 1, 2 \ldots$$

The above equation is equivalent to a quadratic equation in n_1

$$\bar{A}n_1^2 + 2m\sigma n_1 - m^2\sigma = 0$$

or

$$\frac{\omega}{\Omega_1} = k\frac{W_1}{\Omega_1} + m + \frac{m\sigma \pm \sqrt{\sigma\{\sigma + \bar{A}\}}}{\bar{A}} \tag{5.61}$$

where

$$\bar{A} = m^2 + \frac{\sigma^2}{4} + \left[\frac{a\pi}{\log(\frac{R_2}{R_1})}\right]^2 > 0$$

Since the term inside the curly bracket is positive definite, the stability condition depends only on the sign of σ, the density characteristic parameter. Equation (5.61) is exact for $k = o$, saying that negative σ (representing negative density gradient) implies instability, while positive σ (representing positive density gradient) implies stability, which is exactly as predicted in Section (3.3.2), Chapter 3. The corresponding λ in this case is exactly equal to 1/4. For this reason, it is concluded

that the upper bound on λ for which instabilities are possible to exist can not be improved anymore. Besides, it is well known that the solid body rotation for homogeneous fluids (corresponding to $\sigma = 0$) is stable against arbitrary disturbances. The above result clearly reveals that the two-dimensional azimuthal modes are the most unstables ones, and the difference between the onset of instability for axisymmetric modes and non-axisymmetric ones is quite significant.

With the understanding of the stability behavior of the flow in region 1, we now proceed to the analysis of the two-region flow under consideration.

Consider only positive σ for centrifugally stable profile in region 1. The corresponding solutions for perturbation pressure and velocity are previously given in (5.56) and (5.57).

For the profiles in region 2, equation (2.11) can be reduced to

$$D\left[\frac{r^2}{m^2 + k^2 r^2}\left(Du_2 + \frac{u_2}{r}\right)\right] - u_2 = 0 \qquad (5.62)$$

Let

$$u_2 = D\phi$$

Integrating (5.62) and adjusting the integration constant,

one finds

$$D^2\phi + \frac{1}{r}D\phi - (\frac{m^2}{r^2} + k^2)\phi = 0 \qquad (5.63)$$

$$\phi = A_2^* I_m(kr) + B_2^* K_m(kr)$$

or

$$u_2 = N_2[A_2 k^{-(m-1)} I_m'(kr) + B_2 k^{(m+1)} K_m'(kr)] \qquad (5.64)$$

The perturbation pressure in region 2 is obtained by substituting (5.64) into (2.9) and has the form

$$p_2 = -i\rho_2\Omega_2^2 n_2[k\frac{W_2}{\Omega_2} + m(\frac{R}{r})^2 - \frac{\omega}{\Omega_2}][A_2 k^{-m} I_m(kr) + B_2 k^m K_m(kr)] \qquad (5.65)$$

Since the boundary conditions require the perturbation velocity to vanish at R_1 and R_2, the solutions in region 1 and region 2 become

$$\frac{u_1}{N_1} = A_1 k^{-\nu} r^{-\frac{\sigma}{2} - 1}\{[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{qkrI_\nu'(kqr)}{I_\nu(kqr)}]I_\nu(kqr)$$

$$- \hat{H}_1[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{qkrK_\nu'(kqr)}{K_\nu(kqr)}]K_\nu(kqr) \qquad (5.66)$$

$$p_1 = -iA_1 k^{-\nu}\Omega_1^2\rho_1\left(\frac{r}{R}\right)^{\sigma}(n_1^2 - 4 - \sigma)\{r^{-\frac{\sigma}{2}}[I_\nu(kqr) - \hat{H}_1 K_\nu(kqr)]\}$$

$$\frac{u_2}{N_2} = B_2 k^m [kK_m'(kr) - \hat{H}_2 k I_m'(kr)]$$

$$p_2 = -B_2 k^m \rho_2 \Omega_2^2 n_2 \left(k\frac{W_2}{\Omega_2} + m\frac{R^2}{r^2} - \frac{\omega}{\Omega_2}\right)[K_m(kr) - \hat{H}_2 I_m(kr)] \qquad (5.67)$$

where

$$\hat{H}_1 = \frac{\left[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa_1 I_\nu'(q\kappa_1)}{I_\nu(q\kappa_1)}\right]I_\nu(q\kappa_1)}{\left[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa_1 K_\nu'(q\kappa_1)}{K_\nu(q\kappa_1)}\right]K_\nu(q\kappa_1)}$$

$$\hat{H}_2 = \frac{K_m'(\kappa_2)}{I_m'(\kappa_2)}$$

The secular relation for stability characteristics now obtained, by matching both kinematic and dynamic conditions at the interface, is

$$\frac{(n_1^2 - 4 - \sigma)[I_\nu(q\kappa) - \hat{H}_1 K_\nu(q\kappa)]}{\left[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa I_\nu'(q\kappa)}{I_\nu(q\kappa)}\right]I_\nu(qk) - \hat{H}_1\left[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa K_\nu'(q\kappa)}{K_\nu(q\kappa)}\right]K_\nu(q\kappa)}$$

$$- \alpha\beta^2 n_2^2 \frac{K_m(\kappa) - \hat{H}_2 I_m(\kappa)}{\kappa K_m'(\kappa) - \hat{H}_2 \kappa I_m'(\kappa)} = \alpha\beta^2 - 1 - \Gamma_1(1 - m^2 - \kappa^2) \qquad (5.68)$$

α, β, κ and Γ_1 have the same definitions as those in (5.18).

In the absence of boundaries, both $\hat{H}_1$ and $\hat{H}_2$ will go to zero and (5.68) reduces to

$$\frac{n_1^2 - 4 - \sigma}{\left[\frac{2m}{n_1} + \frac{\sigma}{2} + \frac{q\kappa I_\nu'(q\kappa)}{I_\nu(q\kappa)}\right]} - \alpha\beta^2 n_2^2 \frac{K_m(\kappa)}{\kappa K_m'(\kappa)} = \alpha\beta^2 - 1 - \Gamma_1(1 - m^2 - \kappa^2) \qquad (5.69)$$

Uberoi's result can be obtained by letting $\sigma = 0$ and $W_2 = 0$ in the above equation.

Several particular cases of secular relation (5.68) will now be investigated:

5.2.1 Axisymmetric Perturbations (m = 0)

Since $I_\nu'(z) = I_{\nu - 1}(z) - \frac{\nu}{z} I_\nu(z)$

$$K_\nu'(z) = -K_{\nu - 1}(z) - \frac{\nu}{z} K_\nu(z)$$

equation (5.68) for m = 0 can be written as

$$q\hat{F}_1 N_1^2 + \alpha \hat{F}_2 N_2^2 = \kappa\Omega_1^2[\alpha\beta^2 - 1 - \Gamma_1(1 - \kappa^2)] \qquad (5.70)$$

where

$$\hat{F}_1 = \frac{I_\nu(q\kappa)}{I_{\nu-1}(q\kappa)} \; \frac{1 + \dfrac{I_{\nu-1}(q\kappa_1)}{K_{\nu-1}(q\kappa_1)} \; \dfrac{K_\nu(q\kappa)}{I_\nu(q\kappa)}}{1 - \dfrac{I_{\nu-1}(q\kappa_1)}{K_{\nu-1}(q\kappa_1)} \; \dfrac{K_{\nu-1}(q\kappa)}{I_{\nu-1}(q\kappa)}}$$

$$\hat{F}_2 = \frac{K_0(\kappa)}{K_1(\kappa)} \; \frac{1 + \dfrac{K_1(\kappa_2)}{I_1(\kappa_2)} \; \dfrac{I_0(\kappa)}{K_0(\kappa)}}{1 - \dfrac{K_1(\kappa_2)}{I_1(\kappa_2)} \; \dfrac{I_1(\kappa)}{K_1(\kappa)}}$$

If there is no axial velocity difference, i.e., $W_1 = W_2$, equation (5.70) has a simple form

$$(c - W_1)^2 = \frac{R\Omega_1^2[\alpha\beta^2 - 1 - \Gamma_1(1 - \kappa^2)]}{k(q\hat{F}_1 + \alpha\hat{F}_2)} \qquad (5.71)$$

Since $\hat{F}_2$ and the real part of $\hat{F}_1$ are positive definite for $\kappa_1 \leq \kappa \leq \kappa_2$, the stability condition is now determined by the terms inside the square bracket, and the stability boundary for axisymmetric disturbances and zero axial flow gradient is again

$$\alpha\beta^2 - 1 = \Gamma_1(1 - \kappa^2) \qquad (5.72)$$

This relation for zero surface tension is also obtainable

by integrating the Rayleigh-Synge condition across the interface.

5.2.2 Azimuthally Periodic Perturbations ($k = 0$)

For $\sigma = 0$ and small axial wave numbers, equation (5.68) can be written in an algebraic form by the small argument expansion of the modified Bessel functions. The secular relation is reducible to the quadratic equation

$$n_1(n_1 f_1 - 2) + \alpha\beta^2 n_2^2 f_2 = m[\alpha\beta^2 - 1 - \Gamma_1(1 - m^2 - \kappa^2)] \tag{5.73}$$

with f_1 and f_2 defined in (5.29).

Equation (5.73) allows one to solve for ω and yields

$$\omega = \frac{[f_1(kW_1 + m\Omega_1) - \Omega_1] + \alpha f_2(kW_2 + m\Omega_2) \pm \Omega_1\sqrt{\Delta}}{f_1 + \alpha f_2} \tag{5.74}$$

with

$$\Delta = -\alpha f_1 f_2[\gamma_1 + m(\beta - 1) + \frac{1}{f_1}]^2 + (f_1 + \alpha f_2)\{\frac{1}{f_1} + m[\alpha\beta^2 - 1 - \Gamma_1(1 - m^2 - \kappa^2)]\}$$

The stability criterion corresponds to

$$\Delta = 0 \tag{5.75}$$

Several phenomena obtained for special cases follow:

(a) Effects of axial wave numbers

For unbounded flow, the stability boundary without surface tension for k = 0 is

$$\alpha[m(\beta - 1) + 1]^2 - (1 + \alpha)[1 + m(\alpha\beta^2 - 1)] = 0 \tag{5.76}$$

Figure 5.12 shows the stability boundaries for different values of m for k = o. These curves do not intersect each other and those with larger wave numbers correspond to the less stable modes. Such azimuthal modes will amplify in time and the stable region will eventally shrink into the line $\beta = 1$, $\alpha \geq 1$. The lines $\alpha\beta^2 = 1$ corresponding to stability boundaries for axisymmetric modes are plotted and compared to those for azimuthally periodic ones. Once again, it is observed that those stability boundaries lie partially in the region stable to axisymmetric modes in the α - β plane. This means the present flow can be quite stable for axisymmetric perturbations yet unstable to non-axisymmetric ones. These results also show that the Rayleigh-Synge criterion, corresponding to $\alpha\beta^2 = 1$ in the present example, is not valid when non-axisymmetric perturbations are taken into account, and

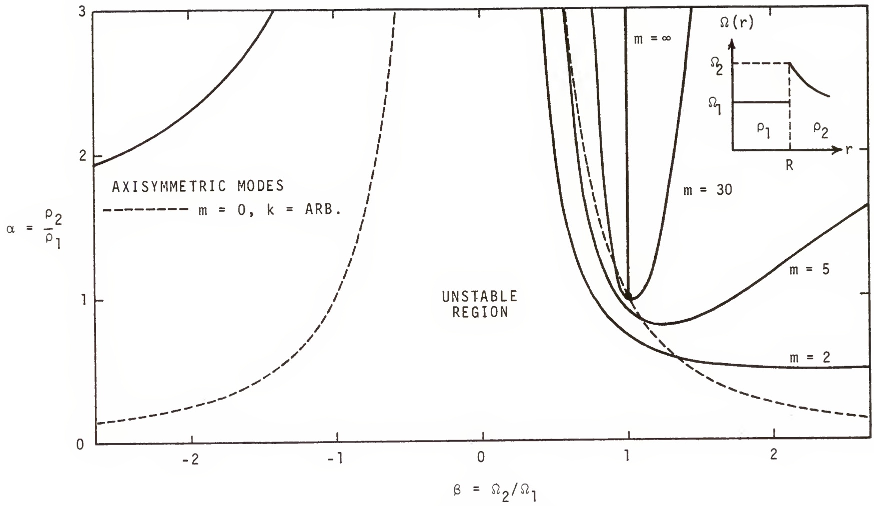


Figure 5.12 Stability boundaries for axis mmetric disturbances and azimuthally periodic disturbances of different values of m.

that two dimensional m modes are most unstable.

(b) <u>Effects of axial flow difference for small k</u>

The influence of large axial flow difference (i.e., $|W_2 - W_1|$ large) is still of importance even for small axial wave numbers. Equation (5.75) without surface tension and solid boundaries takes the form

$$\alpha[\gamma_1 + m(\beta - 1) + 1]^2 - (1 + \alpha)[1 + m(\alpha\beta^2 - 1)] = 0 \quad (5.77)$$

The stability boundaries are plotted in Figure 5.13. The behavior of those curves under the influence of axial flow difference resembles that of the two-region flow with constant angular velocities. Positive γ_1 has destabilizing effects when β is positive and stabilizing effects when β is negative; and vice versa for negative γ_1. Figure 5.14 shows the stability boundaries for different values of m in the $\alpha - \gamma_1$ plane, the angular velocity being continuous across the interface. All the stability boundaries pass through the point $\alpha = 1$, $\gamma_1 = \sqrt{2} - 1$. The decrease of density and the increase of axial flow difference both destabilize the flow.

(c) <u>Effects of boundaries</u>

The stability boundary described by equation (5.75) in the absence of surface tension is

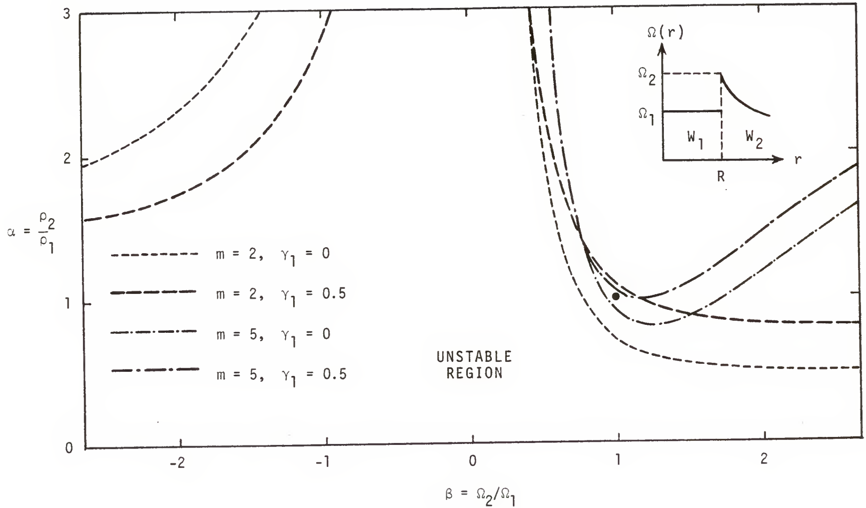


Figure 5.13 Effects of axial flow differences on stability boundaries for small axial wave numbers.

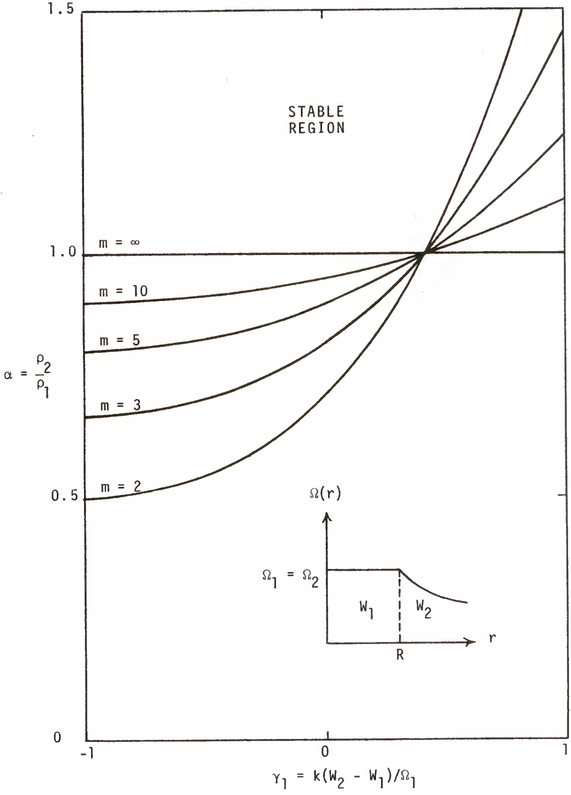


Figure 5.14 Stability boundaries of continuous angular velocity for small axial wave numbers.

$$\alpha f_1 f_2[\frac{1}{f_1} + m(\beta - 1)]^2 - (f_1 + \alpha f_2)[\frac{1}{f_1} + m(\alpha\beta^2 - 1)] = 0 \qquad (5.78)$$

Figure 5.15 reveals that the flow is destabilized by the inward moving walls for $m = 2$. This destabilization, contradicting physical expectations especially when the solid boundaries are sufficiently close, is a result of the assumed discontinuity arising in the velocity field as explained previously in the two-region flow for constant velocities. This phenomenon diminishes as m increases and the stability boundary will be unaffected by the walls for sufficiently short azimuthal waves.

5.2.3 A Non-rotating Jet Core with Density Variations Surrounded by a Homogeneous Potential Vortex

For $\Omega_1 = 0$, equation (5.68) yields

$$\hat{E}_1(kW_1 - \omega)^2 - \alpha\hat{E}_2(kW_2 + m\Omega_2 - \omega)^2 = \alpha\Omega_2^2[1 - \Gamma_2(1 - m^2 - \kappa^2)] \qquad (5.79)$$

where

$$\hat{E}_1 = \frac{I_\nu(\kappa) - \hat{H}_1 K_\nu(\kappa)}{[\frac{\sigma}{2} + \frac{\kappa I_\nu'(\kappa)}{I_\nu(\kappa)}]I_\nu(\kappa) - \hat{H}_1[\frac{\sigma}{2} + \frac{\kappa K_\nu'(\kappa)}{K_\nu(\kappa)}]K_\nu(\kappa)}$$

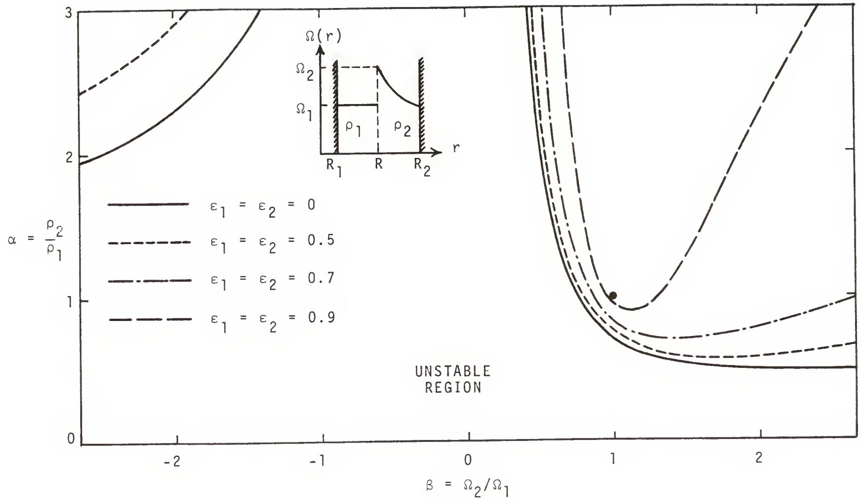


Figure 5.15 Effect of solid walls on the stability boundary for m = 2 subject to azimuthally periodic disturbances.

$$\hat{E}_2 = \frac{K_m(\kappa) - \hat{H}_2 I_m(\kappa)}{\kappa[K_m'(\kappa) - \hat{H}_2 I_m'(\kappa)]}$$

$$\hat{H}_1 = \frac{[\frac{\sigma}{2} + \frac{\kappa_1 I_\nu'(\kappa_1)}{I_\nu(\kappa_1)}] I_\nu(\kappa_1)}{[\frac{\sigma}{2} + \frac{\kappa_1 K_\nu'(\kappa_1)}{K_\nu(\kappa_1)}] K_\nu(\kappa_1)}$$

$$\hat{H}_2 = \frac{K_m'(\kappa_2)}{I_m'(\kappa_2)}$$

$$\nu = \sqrt{m^2 + (\frac{\sigma}{2})^2}$$

$$\Gamma_2 = \frac{T}{\rho_2 \Omega_2^2 R^3}$$

Equation (5.79) is a quadratic equation for ω and can be solved as

$$\omega = \frac{\hat{E}_1(kW_1) - \alpha\hat{E}_2(kW_2 + m\Omega_2)}{\hat{E}_1 - \alpha\hat{E}_2} \pm$$

$$i\frac{\Omega_2\sqrt{-\alpha\hat{E}_1\hat{E}_2(\gamma_2 + m)^2 - \alpha(\hat{E}_1 - \alpha\hat{E}_2)[1 - \Gamma_2(1 - m^2 - \kappa^2)]}}{\hat{E}_1 - \alpha\hat{E}_2}$$

with

$$\gamma_2 = \frac{k(W_2 - W_1)}{\Omega_2} \tag{5.80}$$

The flow will be stable whenever

$$\hat{E}_1\hat{E}_2(\gamma_2 + m)^2 + (\hat{E}_1 - \alpha\hat{E}_2)[1 - \Gamma_2(1 - m^2 - \kappa^2)] \geq 0 \quad (5.81)$$

Since $\hat{E}_1 \geq 0$, $\hat{E}_2 \leq 0$ (see Appendix), both the axial velocity difference and the azimuthal wave numbers have destabilizing effects provided they are of the same sign. Surface tension always stabilizes the flow, except when $m = 0$ and $\kappa < 1$.

For maximum growth rates of the unstable solutions, one differentiates the imaginary part of (5.80) with respect to α and obtains

$$|\omega_i|_{max} = \frac{|\Omega_2(\gamma_2 + m)|}{2}[1 + \frac{1 - \Gamma_2(1 - m^2 - \kappa^2)}{\hat{E}_2(\gamma_2 + m)^2}] \quad (5.82a)$$

with the corresponding real part of ω

$$\omega_r = \frac{1}{2}[kW_1 + kW_2 + m\Omega_2 + \Omega_2\frac{1 - \Gamma_2(1 - m^2 - \kappa^2)}{\hat{E}_2(\gamma_2 + m)}] \quad (5.82b)$$

and density ratio

$$\alpha = -(\frac{\hat{E}_1}{\hat{E}_2})\frac{\hat{E}_2(\gamma_2 + m)^2 + 1 - \Gamma_2(1 - m^2 - \kappa^2)}{\hat{E}_2(\gamma_2 + m)^2 - 1 + \Gamma_2(1 - m^2 - \kappa^2)} \quad (5.82c)$$

It is interesting to note that all the unstable solutions for the flow under consideration, irrespective of the types of perturbations, the boundary effects or the surface tension existing at the interface, lie exactly on a certain semi-circle governed by the equation

$$\{\omega_r - (kW_1 \pm |\omega_i|_{max})\}^2 + \omega_i^2 = |\omega_i|^2_{max} \qquad (5.83)$$

The positive or negative sign corresponds to the positive or negative value of $\{\Omega_2(\gamma_2 + m)\}$ respectively. The locus of maximum growths lies exactly on a straight line in the complex ω plane.

We will now discuss some of the stability characteristics concerning unbounded and bounded flows.

(a) Unbounded flows

For unbounded flows, both $\hat{H}_1$ and $\hat{H}_2$ in (5.79) go to zero, so that the equation yields

$$\frac{(kW_1 - \omega)^2}{\frac{\sigma}{2} + \frac{\kappa I_\nu'(\kappa)}{I_\nu(\kappa)}} - \frac{(kW_2 + m\Omega_2 - \omega)^2}{\frac{K_m(\kappa)}{\kappa K_m'(\kappa)}} = \alpha\Omega_2^2[1 - \Gamma_2(1 - m^2 - \kappa^2)] \qquad (5.84)$$

This flow, in the absence of surface tension, density variations and axial velocities, was originally investigated by Michalke and Timme (1967) for arbitrary disturbances.

They, however, failed to consider the centrifugal effect due to the angular velocity jump at the interface.

In this particular case, the neutral stability condition as deduced from (5.81) is

$$(m+\gamma_2)^2 - \{\alpha[\frac{\sigma}{2} + \frac{\kappa I_\nu'(\kappa)}{I_\nu(\kappa)}] - \frac{\kappa K_m'(\kappa)}{K_m(\kappa)}\} = 0$$

and stability can be guaranteed whenever

$$\alpha \geq \frac{(m+\gamma_2)^2 + \frac{\kappa K_m'(\kappa)}{K_m(\kappa)}}{[\frac{\sigma}{2} + \frac{I_\nu'(\kappa)}{I_\nu(\kappa)}]} \tag{5.85}$$

Figures 5.16 a, b, c show the stability boundaries for different values of σ and m. Both the increase of the relative axial velocity and the decrease of the density ratio at the interface have destabilizing effects. The unstable solutions for $\sigma = 0$ and $m = 0$ are plotted in Figure 5.16d, which are all exactly on a semi-circle governed by the equation

$$\{c_r - (W_1 + |c_i|_{max})\}^2 + c_i^2 = |c_i|^2_{max} \tag{5.86}$$

where

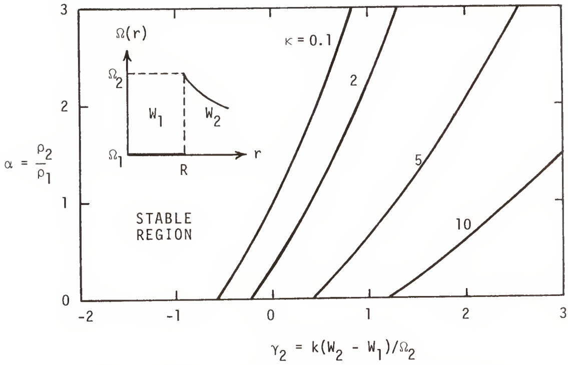


Figure 5.16a Stability boundaries for m = 2 and $\sigma = 0$ (symmetric w.r.t. $\gamma_2 = -2$).

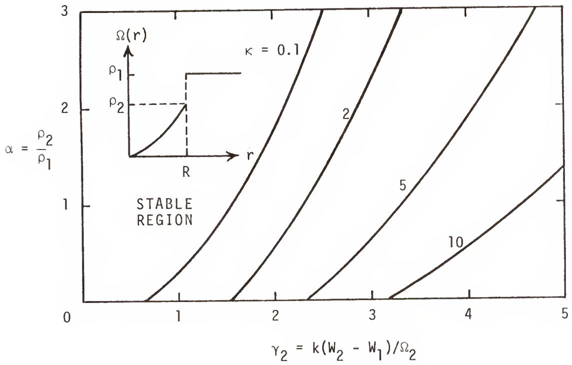


Figure 5.16b Stability boundaries for m = 0 and $\sigma = 2$ (symmetric w.r.t. $\gamma_2 = 0$).

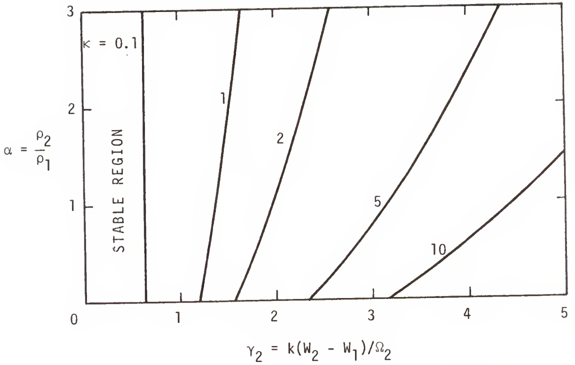


Figure 5.16c Stability boundaries for $m = 0$ and $\sigma = 0$ (symmetric w.r.t. $\gamma_2 = 0$).

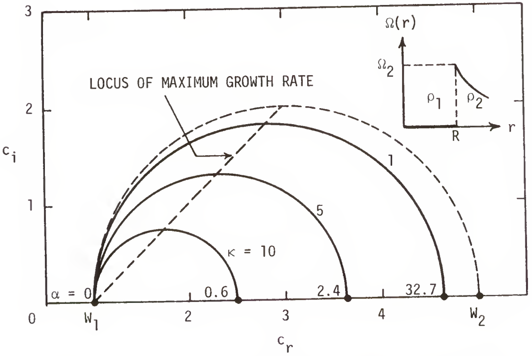


Figure 5.16d Effect of axial wave numbers on the complex wave speed for $m = 0$ and $\sigma = 0$.

$$|c_i|_{max} = \frac{1}{2}|(W_2 - W_1)[1 + \frac{1}{\gamma_2^2} \frac{\kappa K_m'(\kappa)}{K_m(\kappa)}]|$$

with the locus of maximum growth rate on a straight line

$$c_r - c_i = W_1 \tag{5.87}$$

Another result which should be noted is that all the unstable solutions for the axisymmetric case lie inside a semi-circle with the range of the axial velocity as diameter. It gives a solid support to the semi-circle theorem derived in equation (3.31).

Perhaps it should be emphasized once more that the Rayleigh-Synge criterion is no longer valid, even only for axisymmetric disturbances, once the axial velocity gradients are taken into account. Instabilities still take place through the shear layer arising from the axial flow difference at the interface.

(b) Bounded flows

For simplicity the stability boundary described by equation (5.81) for bounded flows can be approximated by the lower and upper limits of axial wave numbers (though there is no difficulty in evaluating the equation for arbitrary k).

For k = 0, the stability boundary is governed by

the equation

$$\alpha = \left(1 - \frac{1}{m\hat{f}_2}\right)\left(\hat{f}_1 \nu - \frac{\sigma}{2}\right) \tag{5.88}$$

where

$$\hat{f}_1 = \frac{1 + \varepsilon_1^{2\nu}}{1 - \varepsilon_1^{2\nu}}$$

$$\hat{f}_2 = \frac{1 + \varepsilon_2^{2m}}{1 - \varepsilon_2^{2m}}$$

Stabilities can be quaranteed when the density ratio at the interface exceeds that of (5.88) for a given value of parameter σ.

Figure 5.17 shows the behavior of the stability boundaries subject to the change of boundary distances. both the density ratio α and the density parameter σ serve as a stabilizing factor to the flow. The destabilizing effect of moving the boundaries inwards increases with descreasing azimuthal wave numbers, and occurs when the distance between two walls is sufficiently close.

For large axial wave numbers, the stability boundary of (5.81) can be written, after some algebraic manipulation, as

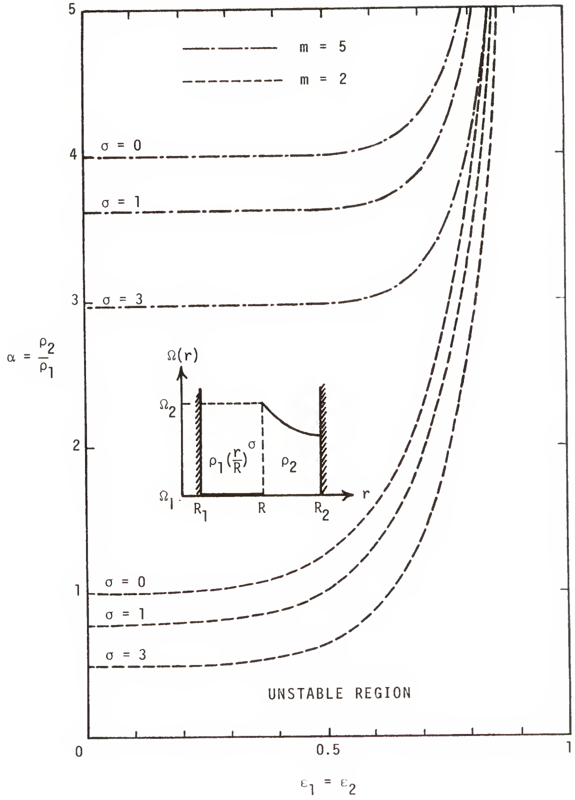


Figure 5.17 Effects of boundary ratios on stability boundaries for azimuthally periodic modes.

$$\alpha = \frac{(\frac{\sigma}{2} - \kappa_0 f_1^*)[(\gamma_2 + m)^2 - \frac{\kappa}{f_2^*}]}{(\frac{\sigma}{2})^2 + \frac{\sigma}{2}(\kappa - \kappa_0)f_1^* - \kappa\kappa_0} \tag{5.89}$$

where

$$f_1^* = \frac{1 + e^{-2(\kappa - \kappa_1)}}{1 - e^{-2(\kappa - \kappa_1)}}$$

$$f_2^* = \frac{1 + e^{-2(\kappa_2 - \kappa)}}{1 - e^{-2(\kappa_2 - \kappa)}}$$

The stability boundaries are plotted in Figure 5.18. The increase of density gradient stabilizes the flow. The destabilization effect of the solid walls in the present case is not that significant when compared with that for $k = 0$, and will not be noticed until the distance between walls is quite close.

5.2.4. Non-rotating Jets with Radial Density Variations

Equation (5.68) for $\Omega_1 = \Omega_2 = 0$ yields

$$\hat{E}_1(W_1 - c)^2 - \alpha\hat{E}_2(W_2 - c)^2 = -\Gamma(1 - m^2 - \kappa^2) \tag{5.90}$$

Recall that

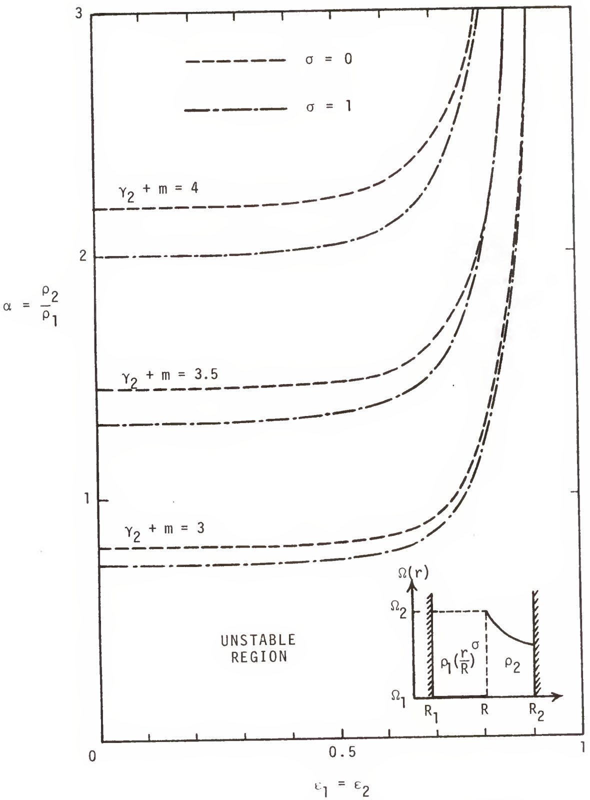


Figure 5.18 Effects of boundary ratios on stability boundaries for $\kappa = 5$.

$$\Gamma = \frac{T}{\rho_1 k^2 R^3}$$

and $\hat{E}_1$ and $\hat{E}_2$ are previously defined in (5.79)

For $\sigma = 0$, $\nu = m$, equation (5.90) reduces exactly to (5.40), i.e., two homogeneous concentric jets with different densities. Since both (5.40) and (5.90) have exactly the same form except with a slightly different definition between E_1 and $\hat{E}_1$ which are both constants, all the analysis and results we have carried earlier can of course be applied here. Therefore, it is concluded that the stability condition for the flow is

$$\alpha\hat{E}_1\hat{E}_2(W_2 - W_1)^2 - (\hat{E}_1 - \alpha\hat{E}_2)\Gamma(1 - m^2 - \kappa^2) \geq 0 \qquad (5.91)$$

which is always violated in the absence of surface tension, despite the solid boundary effects. In addition, all the unstable solutions in the complex c plane lie exactly on a certain semi-circle satisfying the equation

$$\{c_r - (W_2 \pm |c_i|_{max})\}^2 + \{c_i\}^2 = |c_i|^2_{max} \qquad (5.92)$$

with the maximum growth rate lying on the straight line

$$c_i - c_r = \tfrac{1}{2}[|W_1 - W_2| - (W_1 + W_2)]$$

where

$$|c_i|_{max} = \frac{1}{2}[|W_1 - W_2| + \frac{\Gamma_1(1 - m^2 - \kappa^2)}{\hat{E}_1 |W_1 - W_2|}]$$

with the positive or minus sign depending on the sign of $\{W_1 - W_2\}$ respectively.

In the absence of surface tension, all the solutions, which are always unstable, lie exactly on a semicircle in the complex c plane, with the range of axial velocity as diameter.

The density for maximum growth rates ($c_i = \frac{1}{2}|W_1 - W_2|$) lies on the line

$$\alpha = -\frac{\hat{E}_1}{\hat{E}_2} \frac{[\hat{E}_1(W_1 - W_2)^2 - \Gamma_1(1 - m^2 - \kappa^2)]}{[\hat{E}_1(W_1 - W_2)^2 + \Gamma_1(1 - m^2 - \kappa^2)]} \tag{5.93}$$

In the case of zero surface tension, the above expression can be approximated by using expressions valid for small and large axial wave lengths, i.e.,

$$\alpha = \frac{\hat{f}_1 \sqrt{m^2 + (\frac{\sigma}{2})^2} - \frac{\sigma}{2}}{m\hat{f}_2} \qquad \text{for } \kappa << 1 \tag{5.94a}$$

$$\alpha = (\frac{\kappa}{f_2^*}) \frac{\frac{\sigma}{2} - \kappa_1 f_1^*}{(\frac{\sigma}{2})^2 + \frac{\sigma}{2}(\kappa - \kappa_1)f_1^* - \kappa\kappa_1} \qquad \text{for } \kappa >> 1 \tag{5.94b}$$

where $\hat{f}_1$, $\hat{f}_2$, f_1^* and f_2^* are as defined in (5.88) and (5.89). Figures 5.19a,b show the curves of maximum growth rate ($c_i = \frac{1}{2}|W_1 - W_2|$) subject to the change of solid boundaries and the density parameter σ. For sufficiently long or short axial wavelengths with $\sigma = 0$, the curves for maximum growth rate will approach the line $\alpha = 1$.

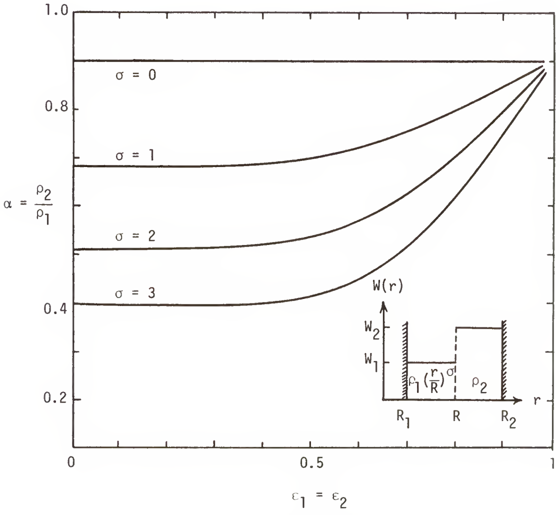


Figure 5.19a Effects of boundary ratios on the curves of maximum growth rate for m = 0 and small k.

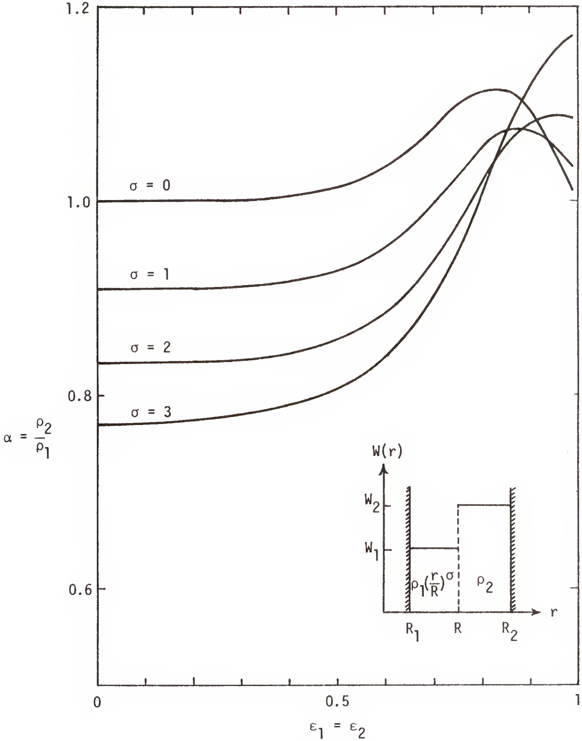
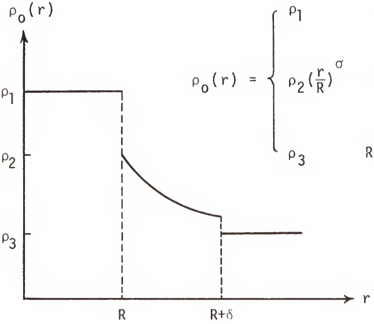


Figure 5.19b Effects of boundary ratios on the curves of maximum growth rate for κ = 5.

5.3 Rigid Body Rotation -- Three-region Solution

Rigid body rotation is of practical interest in the newly developed counterflow centrifuge method for separation of uranium isotopes, and also is one of two velocity profiles for which analytical solutions in terms of well-known functions are possible.

Consider three regions of fluids with the interfaces at R and R + δ in a constant angular and axial velocity field $[0,\ r\Omega_o,\ W_o]$. The density distribution in each region is shown in Figure 5.20.



$$\rho_o(r) = \begin{cases} \rho_1 & 0 \leq r < R \\ \rho_2\left(\frac{r}{R}\right)^{\sigma} & R < r < R+\delta \\ \rho_3 & R+\delta < r < \infty \end{cases}$$

Figure 5.20 Density distribution for rigid body rotation.

Due to the unboundedness of the modified Bessel function of the first kind at infinity and the second kind at the origin, the solutions for the perturbation velocity and pressure in each individual region found in Sections1 and 2 in this chapter are simply

$$\begin{cases} \dfrac{u_1}{N_o} = A_1 k^{-m}\left[\dfrac{\hat{q}krI'_m(qkr)}{I_m(\hat{q}kr)} + \dfrac{2m}{n_o}\right]\dfrac{I_m(\hat{q}kr)}{r} \\ \\ p_1 = -i\rho_1\Omega_o^2(n_o^2 - 4)[A_1k^{-m}I_m(\hat{q}kr)] \end{cases}$$

$$\begin{cases} \dfrac{u_2}{N_o} = r^{-\frac{\sigma}{2} - 1}\left\{A_2k^{-\nu}\left[\dfrac{2m}{n_o} + \dfrac{\sigma}{2} + \dfrac{qkrI'_\nu(qkr)}{I_\nu(qkr)}\right]I_\nu(qkr) + \right. \\ \qquad \left. B_2k^{\nu}\left[\dfrac{2m}{n_o} + \dfrac{\sigma}{2} + \dfrac{qkrK'_\nu(qkr)}{K_\nu(qkr)}\right]K_\nu(qkr)\right\} \\ p_2 = -i\rho_2\left(\dfrac{r}{R}\right)^{\sigma}\Omega_o^2(n_o^2 - 4 - \sigma)\{r^{-\frac{\sigma}{2}}[A_2k^{-\nu}I_\nu(qkr) + B_2k^{\nu}K_\nu(qkr)] \end{cases}$$

$$\begin{cases} \dfrac{u_3}{N_o} = B_3k^m\left[\dfrac{krK'_m(\hat{q}kr)}{K_m(\hat{q}kr)} + \dfrac{2m}{n_o}\right]\dfrac{K_m(\hat{q}kr)}{r} \\ \\ p_3 = -i\rho_3\Omega_o^2(n_o^2 - 4)[B_3k^mK_m(\hat{q}kr)] \end{cases} \qquad (5.95)$$

where

$$\hat{q} = \sqrt{1 - \frac{4}{n_o^2}} \qquad q = \sqrt{1 - \frac{4 + \sigma}{n_o^2}}$$

$$n_o = \frac{N_o}{\Omega_o} = k\frac{W_o}{\Omega_o} + m - \frac{\omega}{\Omega_o}$$

After matching the kinematic and dynamic boundary conditions at both the interfaces, we obtain the secular relation for stability in the absence of surface tension.

$$\frac{\left[\frac{2m}{n_o} + \frac{\sigma}{2} + \frac{q\kappa I'_\nu(q\kappa)}{I_\nu(q\kappa)} - D_1\right]I_\nu(q\kappa)}{\left[\frac{2m}{n_o} + \frac{\sigma}{2} + \frac{q\kappa K'_\nu(q\kappa)}{K_\nu(q\kappa)} - D_1\right]K_\nu(q\kappa)} = \frac{\left[\frac{2m}{n} + \frac{\sigma}{2} + \frac{qhI'_\nu(qh)}{I_\nu(qh)} - D_2\right]I_\nu(qh)}{\left[\frac{2m}{n} + \frac{\sigma}{2} + \frac{qhK'_\nu(qh)}{K_\nu(qh)} - D_2\right]K_\nu(qh)} \tag{5.96}$$

where

$$D_1 = \alpha_1 \frac{n_o^2 - (\sigma + 4)}{(1 - \alpha_1) + \dfrac{n_o^2 - 4}{\dfrac{2m}{n_o} + \dfrac{\hat{q}\kappa I'_m(\hat{q}\kappa)}{I_m(\hat{q}\kappa)}}}$$

$$D_2 = \alpha_2 \frac{n_o^2 - (\sigma + 4)}{(1 - \alpha_2) + \dfrac{n_o^2 - 4}{\dfrac{2m}{n_o} + \dfrac{\hat{q}h K'_m(\hat{q}h)}{K_m(\hat{q}h)}}}$$

$$h = k(R + \delta) = \kappa(1 + \varepsilon) \qquad \varepsilon = \frac{\delta}{R}$$

The density ratios at the interfaces are defined as

$$\alpha_1 = \frac{\rho_2}{\rho_1} \qquad \alpha_2 = \frac{\rho_2(1+\varepsilon)^{\sigma}}{\rho_3} \tag{5.97}$$

From earlier discussions we know that the azimuthally periodic modes play an important role in flow stability. This suggests that we first look at the case of finite m and small axial wave numbers.

5.3.1 Azimuthally Periodic Disturbances (k = 0)

Equation (5.96) for k = 0 yields

$$\frac{(\alpha_1 - 1)\Lambda + m + \alpha_1(\frac{\sigma}{2} - \nu)}{(\alpha_2 - 1)\Lambda - m + \alpha_2(\frac{\sigma}{2} - \nu)} = \frac{(\alpha_1 - 1)\Lambda + m + \alpha_1(\frac{\sigma}{2} + \nu)}{(\alpha_2 - 1)\Lambda - m + \alpha_2(\frac{\sigma}{2} + \nu)}(1+\varepsilon)^{2\nu} \tag{5.98}$$

where Λ has been defined in (3.37a).

Two special cases of density profiles are to be discussed as follows:

(a) Step function type profiles

For $\sigma = 0$, $\nu = m$, equation (5.98) can be rearranged

$$f[(\rho_3 - \rho_2)(\rho_2 - \rho_1)]\Lambda^2 + 2m\rho_2(\rho_3 - \rho_1)\Lambda - m^2[f(\rho_3 - \rho_2)(\rho_2 - \rho_1) + 2\rho_2(\rho_1 + \rho_3)] = 0$$

or

$$\Lambda = m\frac{\rho_2(\rho_1-\rho_3)\pm\sqrt{[f(\rho_3-\rho_2)(\rho_2-\rho_1)+\rho_2(\rho_1-\rho_3)]^2-4f\rho_1\rho_2(\rho_3-\rho_2)(\rho_2-\rho_1)}}{f(\rho_3 - \rho_2)(\rho_2 - \rho_1)} \qquad (5.99)$$

where ok

$$f = 1 - (1 + \varepsilon)^{-2m} \qquad 1 > f \geq 0$$

If $(\rho_3 - \rho_2)(\rho_2 - \rho_1) < 0$, the density distribution either behaves like Figure 5.21a or Figure 5.21b.

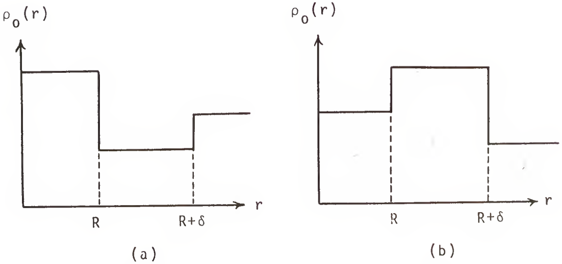


Figure 5.21 Step-type density distribution.

Recall equation (3.39)

$$\frac{\omega}{m\Omega_o} = \frac{\Lambda - 1 \pm \sqrt{1 - \Lambda}}{\Lambda} \qquad (5.100)$$

Since we have proved in Section (3.3.2) that Λ has to be real in the Sturm-Liouville system, instabilities will be expected whenever

$$\Lambda > 1 \tag{5.101}$$

Combining (5.99) and (5.101), and considering only the minus sign in front of the radical in (5.99) for physical reasons, one finds that the condition for instability is

$$(m^2 - 1)[f(\rho_3 - \rho_2)(\rho_2 - \rho_1)]^2 - 2mf\rho_2(\rho_3 - \rho_2)(\rho_2 - \rho_1)[(m - 1)\rho_1 + (m + 1)\rho_3] > 0 \tag{5.102}$$

which is always true for the density distributions presently considered; therefore they are always unstable. This situation is expected if we look at the general criterion for rigid body rotation subject to azimuthally periodic perturbations. Both density profiles in Figures 5.21a,b have a negative density gradient inside the flow region and instabilities would be expected in both cases. If $\rho_1 = \rho_3$, equations (5.99) and (5.100) can be solved for ω with $\alpha = \dfrac{\rho_2}{\rho_1}$

$$\frac{\omega}{\Omega_0} = m - \frac{1 \pm \sqrt{1 - m\hat{f}}}{\hat{f}} \tag{5.103}$$

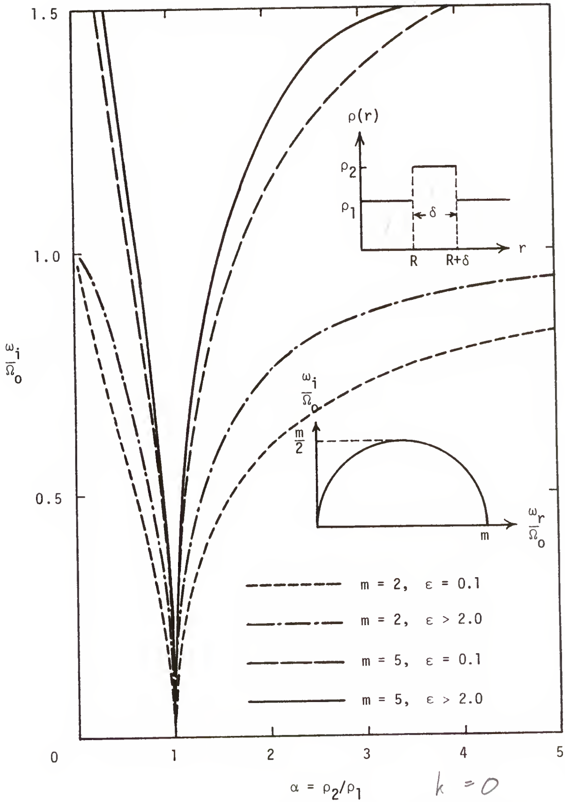
where

$$\hat{f} = \frac{1}{|1 - \alpha|} \sqrt{(1 + \alpha)^2 + \frac{4\alpha}{[(1 + \varepsilon)^{2m} - 1]}}$$

Since $\hat{f}$ is always greater than one for positive values of m, the flow is always unstable except for $\alpha = 1$, i.e., rigid body rotation for homogeneous fluid. The growth rate versus the density ratio between the middle ring of fluid and the surrounding one is plotted in Figure 5.22. The growth rates increase in both directions when the density ratio departs from the line $\alpha = 1$ which is the only stable region for all modes in the $\omega_i - \alpha$ plane. The higher azimuthal wave numbers correspond to the less stable modes; the narrower the middle ring of fluid becomes the lower the growth rates are. When the density of the fluid in the middle region is sufficiently large or small in relation to that of the surrounding one, the flow will be most unstable and possess very large growth rates.

If $(\rho_3 - \rho_2)(\rho_2 - \rho_1) = 0$ (either $\rho_1 = \rho_2$ or $\rho_2 = \rho_3$), the density distribution reduces to a two-region type of profile, and equations (5.99) and (5.100) can be solved for ω, i.e.,

$$\frac{\omega}{\Omega_o} = m - \frac{1 - \alpha \pm \sqrt{(\alpha - 1)[(m + 1)\alpha + (m - 1)]}}{(1 + \alpha)} \qquad (5.104)$$



$k = 0$

$\varepsilon = \frac{\delta}{R}$

Figure 5.22 Growth rates for azimuthally periodic modes.

Ω = same

Unstable modes will take place whenever $\alpha < 1$. This result is also deducible from (5.33) for $\beta = 1$. Note that the unstable solutions for the complex ω in (5.103) and (5.104) lie exactly on a semi-circle governed by the equation

$$\left(\frac{\omega_r}{\Omega_o} - \frac{m}{2}\right)^2 + \left(\frac{\omega_i}{\Omega_o}\right)^2 = \left(\frac{m}{2}\right)^2$$

This instability characteristic has been previously discussed in the general criteria in Section (3.3.2) for rigid body rotation subject to angularly periodic perturbations.

(b) <u>Continuous density distributions</u>

If $\alpha_1 = \alpha_2 = 1$, the density will be continuous across the interface. Equation (5.98) reduces to

$$(2m^2 + \sigma\Lambda - 2m\nu) - (2m^2 + \sigma\Lambda + 2m\nu)(1 + \varepsilon)^{2\nu} = 0 \qquad (5.105)$$

If the middle region is narrow enough, such that $\varepsilon << 1$ and $(1 + \varepsilon)^{2\nu} \simeq 1 + 2\nu\varepsilon$, equation (5.105) yields

$$\frac{\omega}{\Omega_o} = m + \varepsilon\left\{\frac{\sigma \pm \sqrt{\sigma[\sigma + m(2m + \frac{1}{\varepsilon})]}}{1 + 2m\varepsilon}\right\} \qquad (5.106)$$

For positive values of σ, the density profile is an

increasing function of radius and the flow is stable. For negative values of σ, equation (5.106) reveals that the wave velocity will be complex, and hence that the flow is unstable. Figure 5.23 shows the complex wave speed versus the density factor σ for $\varepsilon = 0.1$. The smaller σ is, the higher the growth rate, and the slower the unstable waves will propagate. The complex wave velocity lies exactly on a semi-circle as forecast in the general criteria for rigid body rotation subject to azimuthally periodic perturbations. For finite values of σ, the solutions for the complex ω lie exactly on the right-hand side of the semi-circle as shown in Figure 5.23.

5.3.2 Arbitrary Disturbances for Constant Density in the Middle Region

Equation (5.96) for $\sigma = 0$ can be written as

$$\left[\frac{\frac{2m}{n_o} + \frac{q\kappa K_m'(q\kappa)}{K_m(q\kappa)}}{\frac{2m}{n_o} + \frac{q\kappa I_m'(q\kappa)}{I_m(q\kappa)}} - \alpha_1 \frac{(n_o^2 - 4) + \frac{2m}{n_o} + \frac{q\kappa K_m'(q\kappa)}{K_m(q\kappa)}}{(n_o^2 - 4) + \frac{2m}{n_o} + \frac{q\kappa I_m'(q\kappa)}{I_m(q\kappa)}}\right]\left[\frac{\frac{2m}{n_o} + \frac{qhI_m'(qh)}{I_m(qh)}}{\frac{2m}{n_o} + \frac{qhK_m'(qh)}{K_m(qh)}} -\right.$$

$$\left.\alpha_2 \frac{(n_o^2 - 4) + \frac{2m}{n_o} + \frac{qhI_m'(qh)}{I_m(qh)}}{(n_o^2 - 4) + \frac{2m}{n_o} + \frac{qhK_m'(qh)}{K_m(qh)}}\right] - (1 - \alpha_1)(1 - \alpha_2)\frac{I_m(q\kappa)}{I_m(qh)}\frac{K_m(qh)}{K_m(q\kappa)} = 0 \tag{5.107}$$

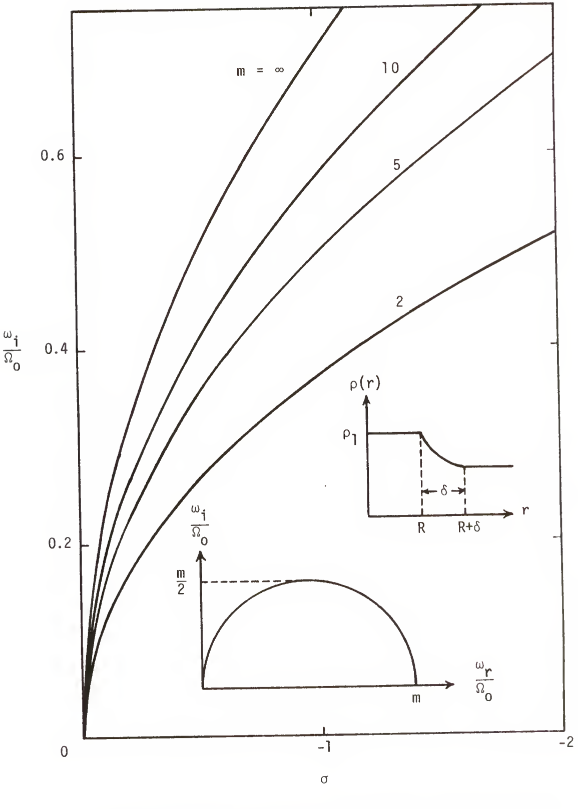


Figure 5.23 Growth rates for azimuthally periodic perturbations (ϵ = 0.1).

with

$$q = \hat{q} = \sqrt{1 - \frac{4}{n_o^2}}$$

The stability of this three-region flow for rigid body rotation in the lower limit of the axial wave numbers has just been discussed. In the following we will investigate the stability behavior of the present flow for large axial wave numbers.

Equation (5.107) for short axial waves can be expanded according to the asymptotic forms of the modified Bessel functions to yield

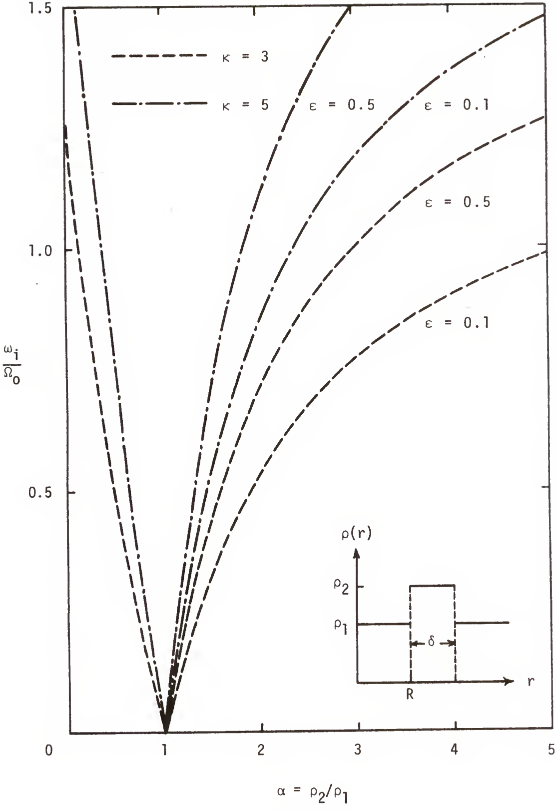
$$\{n_o^2 - [2 \pm \sqrt{4 + \kappa^2(\frac{1 - \alpha_1}{1 + \alpha_1})^2}]\}\{n_o^2 - [2 \pm \sqrt{4 + n^2(\frac{1 - \alpha_2}{1 - \alpha_2})^2}]\} = 0 \tag{5.108}$$

The stability condition depends on the density ratio at the corresponding interface, and the stability boundaries are independent of the azimuthal wave numbers, since only the unstable modes for short axial wave lengths are under consideration. In fact, if the wave lengths are short compared to the thickness of the middle ring of the fluid, instability phenomena will be confined to the neighborhood of the interfaces and the flow will not be affected when it is sufficiently far away from the interfaces. Figure

5.24 shows the growth rate as a function of density ratio for $\alpha_1 = \alpha_2 = \alpha$, i.e., $\rho_1 = \rho_3$. Since instabilities take place at the interface $r = R$ only when $\alpha < 1$ and at the interface $r = R + \delta$ only when $\alpha > 1$, the conditions governing instability following from (5.108) are

$$\left(\frac{\omega_i}{\Omega_o}\right)^2 = \begin{cases} \sqrt{4 + \left[\kappa \frac{1-\alpha}{1+\alpha}\right]^2} - 2 & \text{for } \alpha < 1 \\ \sqrt{4 + \left[\kappa(1+\varepsilon)\frac{1-\alpha}{1+\alpha}\right]^2} - 2 & \text{for } \alpha > 1 \end{cases} \tag{5.109}$$

The behavior of the growth rates in this case resembles that of azimuthally periodic ones. The growth rates for $\alpha > 1$ increase with increasing axial wave numbers as well as the increasing width of the middle ring of the fluid. The unbalanced centrifugal pressure produced by the middle region directly affects the onset of instabilities. The growth rates for $\alpha < 1$ depend only on the centrifugal pressure balance at the interface $r = R$, since only short wave lengths are considered. These unstable modes will exist for all values of α except along the line $\alpha = 1$, i.e., solid body rotation for homogeneous fluid, which is stable against any disturbances.



$k \neq 0$

Figure 5.24 Growth rates for large axial wave numbers (independent of m)

5.4 Non-rotating Jets -- Three-region Solution

We know that the instability mechanism for non-rotating jets is a purely shear effect between two adjacent fluid layers, and that the particle exchange involves no potential energy change since the gravitational effects of the fluid are neglected. The sufficient condition for stability in (3.19b) fails to yield any information because of the absence of stabilizing effects existing in the flow field. On the other hand, the governing stability equation for non-rotating jets resembles Rayleigh's equation and yields only a weak necessary condition for instability. As a result of these difficulties, neither stabilities nor instabilities can be predicted in advance. However, unstable solutions would be expected if discontinuities exist in the axial velocity field. The kinetic energy carried by the fluid particles in the neighborhood of the discontinuous sheet will suddenly have a sharp change and easily achieve the particle exchange due to the absence of stabilizing factors. We now construct two examples to demonstrate the instability characteristics of the non-rotating jet flows.

5.4.1 Three Concentric Constant Axial Jets

The first example for non-rotating jets we would

like to investigate is a three-region step-type velocity and density profile with the interfaces at $r = R$ and $r = R + \delta$, i.e.,

$$W(r) = \begin{cases} W_1 \\ W_2 \\ W_3 \end{cases} \qquad \rho_o(r) = \begin{cases} \rho_1 & \text{for } 0 \leq r < R \\ \rho_2 & \text{for } R < r < R + \delta \\ \rho_3 & \text{for } R + \delta < r < \infty \end{cases}$$

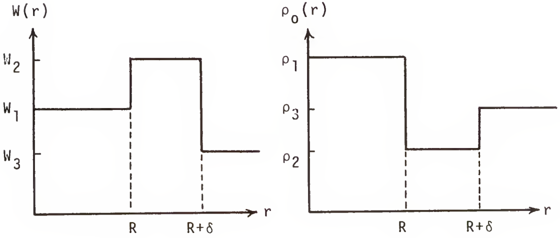


Figure 5.25 Step-type distributions for three concentric jets.

Equation (2.11) for zero azimuthal velocities yields

$$D(\rho_o E D^* u) - [\rho_o k^2 + \frac{D_*(\rho_o EDW)}{W - c}]u = 0 \qquad (5.110)$$

For constant axial velocities and densities, equation (5.110) becomes

$$D[(\frac{r^2}{m^2+k^2r^2})D^*u] - u = 0 \tag{5.111}$$

Letting $u = D\phi$, integrating (5.111) and adjusting the integration constant, we obtain

$$\begin{aligned} &D^2\phi + \frac{1}{r}D\phi - (\frac{m^2}{r^2} + k^2)\phi = 0 \\ &\phi = A^* I_m(kr) + B^* K_m(kr) \end{aligned} \tag{5.112}$$

The solutions for the perturbation velocity and pressure are

$$\frac{u}{N} = AkI_m'(kr) + BkK_m'(kr)$$

$$p = -i\rho_o N^2[AI_m(kr) + BK_m(kr)]$$

or in the individual region we have

$$\begin{cases} \dfrac{u_1}{N_1} = A_1 kI_m'(kr) \\ p_1 = -i\rho_1 N_1^2[A_1 I_m(kr)] \end{cases}$$

$$\begin{cases} \dfrac{u_2}{N_2} = A_2 k I_m'(kr) + B_2 k K_m'(kr) \\ p_2 = -i\rho_2 N_2^2 [A_2 I_m(kr) + B_2 K_m(kr)] \end{cases} \tag{5.113}$$

$$\begin{cases} \dfrac{u_3}{N_3} = B_3 k K_m'(kr) \\ p_3 = -i\rho_3 N_3^2 [B_3 K_m(kr)] \end{cases}$$

After matching both the kinematic and dynamic boundary conditions at the interfaces, the secular relation for stability yields

$$(\rho_2 N_2^2 - \rho_1 N_1^2)(\rho_3 N_3^2 - \rho_2 N_2^2) + [\rho_2 N_2^2 \frac{K_m(\kappa)}{I_m(\kappa)} - \rho_1 N_1^2 \frac{K_m'(\kappa)}{I_m'(\kappa)}][\rho_2 N_2^2 \frac{I_m(h)}{K_m(h)} - \rho_3 N_3^2 \frac{I_m'(h)}{K_m'(h)}] = 0 \tag{5.114}$$

This expression can be reduced to that of the two-region flow by letting either $\rho_1 = \rho_2$ and $W_1 = W_2$, or $\rho_2 = \rho_3$ and $W_2 = W_3$.

Considering $\rho_1 = \rho_3$, $W_1 = W_3$, i.e., an annular jet injected into a solid jet of different density, we expand (5.114) according to the asymptotic forms of the modified Bessel functions and obtain an algebraic relation

$$[\alpha(W_2 - c)^2 - (W_1 - c)^2]^2 - e^{2k\varepsilon}[\alpha(W_2 - c)^2 + (W_1 - c)^2]^2 = 0 \tag{5.115}$$

Equation (5.115) can be solved for c

$$\frac{c}{W_1} = \frac{\alpha(1 + e^{\kappa\varepsilon})\tau - (1 - e^{\kappa\varepsilon}) \pm i(1 - \tau)\sqrt{\alpha(e^{2\kappa\varepsilon} - 1)}}{\alpha(1 + e^{\kappa\varepsilon}) - (1 - e^{\kappa\varepsilon})} \tag{5.116a}$$

$$\frac{c}{W_1} = \frac{\alpha(1 - e^{\kappa\varepsilon})\tau - (1 + e^{\kappa\varepsilon}) \pm i(1 - \tau)\sqrt{\alpha(e^{2\kappa\varepsilon} - 1)}}{\alpha(1 - e^{\kappa\varepsilon}) - (1 + e^{\kappa\varepsilon})} \tag{5.116b}$$

where $\tau = \frac{W_2}{W_1} = \frac{W_2}{W_3}$, $\varepsilon = \frac{\delta}{R}$, and ε is not necessarily small.

Equations (5.116a,b) represent unstable solutions existing at the interfaces $r = R$ and $r = R + \delta$ respectively. For sufficiently short wave lengths, instabilities will take place at both the interfaces for $\tau \neq 1$, and they are not expected to couple with each other. It is interesting to note again all the unstable solutions lie exactly on a semi-circle with the reason given earlier in the discussion of non-rotating jets of the first example.

Figure 5.26 shows the maximum growth rates $c_i = \frac{1}{2}|W_2 - W_1|$ in the $\alpha - \varepsilon$ plane. The curves in the lower part of the plane represent the solutions for maximum growth rate at interface $r = R$, while the curves in the

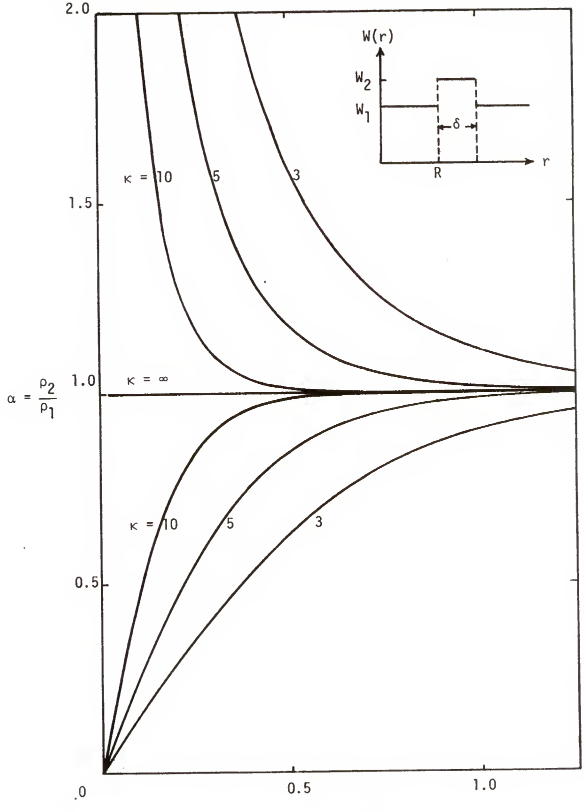


Figure 5.26 Curves of maximum growth rate ($c_i = |W_2 - W_1|/2$)

upper part of the plane represent those at the interface $r = R + \delta$. They both approach the asymptote $\alpha = 1$ which is the most unstable situation. Both the small or large density ratios have stabilizing effects and the flow will be stable for $\alpha = 0$ or $\alpha = \infty$, i.e., an annular liquid jet directed into vacuum or an annular air jet blowing into a heavy fluid is of course stable in the absence of surface tension.

5.4.2 Continuous Axial Jets

The second example of jet instability we wish to discuss is a three-region flow with continuous velocity distributions subject to axisymmetric disturbances.

Squire (1933) (see also Yih 1955) has proven that plane parallel flows are most unstable to two-dimensional disturbances. It is suggested, but not proven, by many analytical and experimental results that the two-dimensional azimuthally periodic perturbations are amplified more strongly than the corresponding three-dimensional ones. In fact, the phase velocities of the perturbations are in general expected to follow the direction of the mean flow. For this reason, the onset of instability for axisymmetric perturbations would be expected to take place in the axisymmetric jet case, and should therefore be discussed.

Consider a velocity and density profile of the following form

$$W = \bar{A}\left(\frac{r}{R}\right)^{2-\sigma} + \bar{B}$$

$$\rho_o = \rho_2\left(\frac{r}{R}\right)^{\sigma} \tag{5.117}$$

here ρ_2, $\bar{A}$, $\bar{B}$, and σ are arbitrary real constants. For axisymmetric disturbances (m = 0), the governing equation for profile (5.117) taken from equation (5.110) is

$$D(r^{\sigma}D^{*}u) - k^2r^{\sigma}u = 0 \tag{5.118}$$

We normalize the equation by setting $u = r^{-\frac{\sigma}{2}}\phi$ and obtain

$$D^2\phi + \frac{1}{r}D\phi - \left[\frac{(1-\frac{\sigma}{2})^2}{r^2} + k^2\right]\phi = 0 \tag{5.119}$$

with solutions in terms of the modified Bessel functions of the first and second kinds of order μ, namely

$$\phi = \hat{A}I_{\mu}(kr) + \hat{B}K_{\mu}(kr)$$

where

$$\mu = \left|1 - \frac{\sigma}{2}\right|$$

The perturbation velocity is

$$u = r^{-\frac{\sigma}{2}}[\hat{A}I_\mu(kr) + \hat{B}K_\mu(kr)] \tag{5.120}$$

Following Michalke and Schade (1963), we construct a three-region flow with velocity and density distribution as follows:

$$W = \begin{cases} W_1 & 0 \leq r \leq R_1 \\ W_1 \dfrac{R_2^{2-\sigma} - r^{2-\sigma}}{R_2^{2-\sigma} - R_1^{2-\sigma}} + W_2 \dfrac{r^{2-\sigma} - R_1^{2-\sigma}}{R_2^{2-\sigma} - R_1^{2-\sigma}} & R_1 \leq r \leq R_2 \\ W_3 & R_2 \leq r < \infty \end{cases}$$

$$\rho_o = \begin{cases} \rho_1 & 0 \leq r < R_1 \\ \rho_2 \left(\dfrac{r}{R_2}\right)^\sigma & R_1 < r < R_2 \\ \rho_3 & R_2 < r < \infty \end{cases}$$

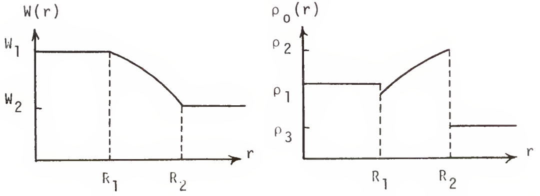


Figure 5.27 Axial velocity and density distributions.

After taking care of the irregularities of the modified Bessel functions, the solutions for the fluctuation velocities and pressures in each region are

$$
\begin{cases}
u_1 = A_1 k I_o'(kr) \\
p_1 = -iA_1[\rho_1 N_1 I_o(kr)]
\end{cases}
$$

$$
\begin{cases}
u_2 = r^{-\frac{\sigma}{2}}[A_2 I_\mu(kr) + B_2 K_\mu(kr)] \\
p_2 = i\frac{\rho_2}{k^2}\left(\frac{r}{R_2}\right)^{\sigma} r^{-\frac{\sigma}{2} - 1}\{A_2\{k\frac{(2 - \sigma)(W_2 - W_1)}{R_2^{2 - \sigma} - R_1^{2 - \sigma}}r^{2 - \sigma} - N[(1 - \frac{\sigma}{2}) + \frac{krI_\mu'(kr)}{I_\mu(kr)}]\}I_\mu(kr) + B_2\{k\frac{(2 - \sigma)(W_2 - W_1)}{R_2^{2 - \sigma} - R_1^{2 - \sigma}} r^{2 - \sigma} - N[(1 - \frac{\sigma}{2}) + \frac{krK_\mu'(kr)}{K_\mu(kr)}]\}K_\mu(kr)
\end{cases}
\quad (5.121)
$$

$$
\begin{cases}
u_3 = B_3 k K_o'(kr) \\
p_3 = -iB_3[\rho_3 N_3 K_o(kr)]
\end{cases}
$$

Matching the interfacial condition at the interfaces, one finds the secular relation governing stability is

$$\begin{vmatrix} [F(\kappa_1)(W_1 - c) + R_1^{2-\sigma}H]I_\mu(\kappa_1) & [G(\kappa_1)(W_1 - c) + R_1^{2-\sigma}H]K_\mu(\kappa_1) \\ [F(\kappa_2)(W_2 - c) + R_2^{2-\sigma}H]I_\mu(\kappa_2) & [G(\kappa_2)(W_2 - c) + R_2^{2-\sigma}H]K_\mu(\kappa_2) \end{vmatrix} = 0 \tag{5.122}$$

where

$$F(\kappa_1) = \frac{\kappa_1}{\alpha_1}\frac{I_o(\kappa_1)}{I_o'(\kappa_1)} - (1 - \frac{\sigma}{2}) - \frac{\kappa_1 I_\mu'(\kappa_1)}{I_\mu(\kappa_1)}$$

$$F(\kappa_2) = \frac{\kappa_2}{\alpha_2}\frac{K_o(\kappa_2)}{K_o'(\kappa_2)} - (1 - \frac{\sigma}{2}) - \frac{\kappa_2 I_\mu'(\kappa_2)}{I_\mu(\kappa_2)}$$

$$G(\kappa_1) = \frac{\kappa_1}{\alpha_1}\frac{I_o(\kappa_1)}{I_o'(\kappa_1)} - (1 - \frac{\sigma}{2}) - \frac{\kappa_1 K_\mu'(\kappa_1)}{K_\mu(\kappa_1)}$$

$$G(\kappa_2) = \frac{\kappa_2}{\alpha_2}\frac{K_o(\kappa_2)}{K_o'(\kappa_2)} - (1 - \frac{\sigma}{2}) - \frac{\kappa_2 K_\mu'(\kappa_2)}{K_\mu(\kappa_2)}$$

$$H = \frac{(2 - \sigma)(W_2 - W_1)}{R_2^{2-\sigma} - R_1^{2-\sigma}}$$

$$\alpha_1 = \frac{\rho_2}{\rho_1}\left(\frac{R_1}{R_2}\right)^{\sigma} \qquad \alpha_2 = \frac{\rho_2}{\rho_3}$$

Equation (5.122) is a quadratic equation for the eigen wave speed c

$$Ac^2 - Bc + D = 0$$

or

$$c = \frac{B \pm \sqrt{B^2 - 4AD}}{2A} \tag{5.123}$$

where

$$A = F(\kappa_1)G(\kappa_2)I_\mu(\kappa_1)K_\mu(\kappa_2) - F(\kappa_2)G(\kappa_1)I_\mu(\kappa_2)K_\mu(\kappa_1)$$

$$B = \{F(\kappa_1)[W_1G(\kappa_2) + HR_2^{2-\sigma}] + G(\kappa_2)[W_2F(\kappa_1) + HR_1^{2-\sigma}]\}I_\mu(\kappa_1)K_\mu(\kappa_2)$$

$$- \{F(\kappa_2)[W_1G(\kappa_1) + HR_1^{2-\sigma}] + G(\kappa_1)[W_2F(\kappa_2) + HR_2^{2-\sigma}]\}I_\mu(\kappa_2)K_\mu(\kappa_1)$$

$$D = [W_1F(\kappa_1) + HR_1^{2-\sigma}][W_2G(\kappa_2) + HR_2^{2-\sigma}]I_\mu(\kappa_1)K_\mu(\kappa_2)$$

$$- [W_1G(\kappa_1) + HR_1^{2-\sigma}][W_2F(\kappa_2) + HR_2^{2-\sigma}]I_\mu(\kappa_2)K_\mu(\kappa_1)$$

Unstable solutions exist whenever

$$B^2 - 4AD < 0$$

This condition reduces to Michalke and Schade's (1963) result for homogeneous fluid (σ = o) and $W_2 = 0$. They obtained some special unstable solutions for long axial wave lengths.

If we expand the modified Bessel functions according to their asymptotic forms and assume wave lengths are

sufficiently small, such that the terms involving $e^{-2(\kappa_2 - \kappa_1)}$ are negligible, then equation (5.123) can be simplified into

$$F(\kappa_2)G(\kappa_1)c^2 - \{F(\kappa_2)[W_1G(\kappa_1) + HR_1^{2-\sigma}] + G(\kappa_1)[W_2F(\kappa_2) + HR_2^{2-\sigma}]\}c$$

$$+ [W_1G(\kappa_1) + HR_1^{2-\sigma}][W_2F(\kappa_2) + HR_2^{2-\sigma}] = 0 \qquad (5.124)$$

In this case, the terms inside the radical are given as

$$B^2 - 4AD = (W_1 - W_2)^2\{F(\kappa_2)G(\kappa_1) + (2-\sigma)[\frac{G(\kappa_1)}{1 - (\frac{R_1}{R_2})^{2-\sigma}} - \frac{F(\kappa_2)}{(\frac{R_2}{R_1})^{2-\sigma} - 1}]\}^2 \geq 0 \qquad (5.125)$$

Therefore the flow will always be stable for sufficiently short waves. This result is expected since the axial velocity under consideration is continuous across both interfaces, and therefore, the flow is stable for large axial wave numbers. The density variation in this case does not have significant influence on the stability characteristics because of neglect of gravitational effects.

CHAPTER 6

CONCLUSION

As shown in the preceding chapters both via some general stability criteria and by exact solutions for several different velocity and density profiles, the instability mechanisms present in heterogeneous swirling flows may be of either centrifugal or shear origin. The classical Rayleigh-Synge criterion is found to be neither a necessary nor a sufficient condition for the stability of such flows when axial shearing rates and/or non-axisymmetric disturbances are considered. A general criterion based on a non-dimensional local number, reminiscent of the Richardson number encountered in stratified parallel flows, has been derived. It is a sufficient condition for stability, subject to infinitesimal arbitrary perturbations, and is found to never be violated for the specific examples considered in Chapter 5. Such a sufficiency condition depends completely on the existing stabilizing effects, which appear mathematically as a second order singularity of the differential equation governing stability. The vanishing of such a singularity (i.e., the vanishing of the stabilizing factors existing in the flow) results, automatically, in the violation of the

sufficiency condition and no stabilities can then be predicted (the same argument can also be applied to the Taylor-Goldstein equation for two-dimensional stratified shear flows). For this reason, no stabilities can be guaranteed for flows governed by stability equations without a second order singularity, such as non-rotating jets or homogeneous two-dimensional parallel flows (i.e., Rayleigh's equation), before solving the governing stability equations.

For possible instabilities in heterogeneous swirling flows, the semi-circle theorem has been verified to be valid for four types of shearing flows possessing centrifugally stable profiles. Unstable solutions for non-rotating jets, rotating flows with axial velocity gradients subject to axisymmetric disturbances, rotating flows for azimuthally periodic disturbances in addition to $D(\rho_0\Omega) = 0$, and swirling flows for the narrow gap approximation must lie inside a certain semi-circle in the complex phase velocity plane. The range between maximum and minimum velocity of the steady-state flow serves as the semi-circle diameter. Another type of semi-circle bound, proved only for rigid body rotation subject to two-dimensional azimuthally periodic perturbations, states that all unstable solutions must lie exactly on a semi-circle in the complex phase velocity plane, with the constant angular velocity as diameter. Several exact analytical solutions have been studied and

the results have been brought up and compared with predictions by the semi-circle theorem. It is interesting to note that the unstable solution for a potential vortex surrounding a non-rotating jet with smoothly varying density distribution and subject to arbitrary perturbations also lies exactly on a certain semi-circle in the complex c plane. The velocity and density profilesfor this case however do not fall into any of the categories of semi-circle bound mentioned above. We believe that the flow belongs basically to the non-rotating jet type, but with a superposed potential vortex which has stabilizing effects on the instability. The reason for an exact semi-circle in this case is mainly due to the constant axial velocity distribution inside each individual region as mentioned before.

It is suggested, but not proven, by the analysis and results that rotating flows with discontinuous velocity profiles are less stable with respect to two-dimensional azimuthally periodic disturbances than to the corresponding three-dimensional ones. Shear effects in this case play a dominant role in the onset of instability. The corresponding most critical perturbations for non-rotating jet flows are axisymmetric ones.

For broken-line profiles with velocity or density discontinuities at the interfaces, the centrifugal effects must always be included in the dynamic interfacial conditions.

Failure to do this led Michalke and Timme (1967) to perform an erroneous calculation at the cylindrical vortex sheet.

The following briefly summarize the results on the stability behavior of the heterogeneous swirling flows obtained in this study:

1. There are two types of instability mechanisms for heterogeneous swirling flows: centrifugal and shear. The Rayleigh-Synge criterion is only a condition for centrifugal stability.
2. The sufficient condition for stability is $\lambda \geq \frac{1}{4}$ [refer to equation (3.19) for definition] everywhere inside the flow domain.
3. The semi-circle theorem holds for four types of centrifugally stable profiles. Unstable solutions for rigid body rotation against azimuthally periodic perturbations must lie exactly on a semi-circle in the complex eigen velocity plane.
4. Densities increasing radially outwards have stabilizing effects. This has been noted in a related study by Kurzweg (1969).
5. Axial velocity gradients, of eigher sign, destabilize the flow. This is a manifestation of the familiar Kelvin-Helmholtz instability mechanism.
6. Azimuthally periodic disturbances are most unstable for rotating shearing flows.

7. The centrifugal effects due to the discontinuity of density or azimuthal velocity must be included in the dynamic interfacial condition.
8. Surface tension always stabilizes the flow except for $m = 0$ when $\kappa < 1$.
9. The phase velocities of perturbations are in general expected to follow the direction of the mean flow, with the larger wave numbers corresponding to stronger amplified growths for vortex sheets existing in the flow.
10. Unstable solutions for rigid body rotation with constant axial flow and density distribution $\rho_0(r) \sim r^{\sigma}$ subject to arbitrary perturbations are solved exactly in terms of the modified Bessel functions with complex auguments and orders. For azimuthally periodic perturbations instability occurs when $\sigma < 0$, while for axisymmetric perturbations instability will not take place until $\sigma < -4$.
11. Non-rotating jet flows with axial velocity $W(r) = Ar^{2-\sigma} + B$ and density $\rho_0(r) \sim r^{\sigma}$ are also solved in terms of the modified Bessel functions with real arguments and orders. This profile is drawn between an inner and an outer concentric jet of constant density. The flow is found to be always stable for sufficiently short axial wave lengths.

APPENDIX

$$\hat{E}_1 = \frac{1 - \dfrac{[\frac{\sigma}{2} + \frac{\kappa_1 I_\nu'(\kappa_1)}{I_\nu(\kappa_1)}]\, I_\nu(\kappa_1)K_\nu(\kappa)}{[\frac{\sigma}{2} + \frac{\kappa_1 K_\nu'(\kappa_1)}{K_\nu(\kappa_1)}]\, K_\nu(\kappa_1)I_\nu(\kappa)}}{1 - \dfrac{[\frac{\sigma}{2} + \frac{\kappa_1 I_\nu'(\kappa_1)}{I_\nu(\kappa_1)}][\frac{\sigma}{2} + \frac{\kappa K_\nu'(\kappa)}{K_\nu(\kappa)}]\, I_\nu(\kappa_1)K_\nu(\kappa)}{[\frac{\sigma}{2} + \frac{\kappa_1 K_\nu'(\kappa_1)}{K_\nu(\kappa_1)}][\frac{\sigma}{2} + \frac{\kappa I_\nu'(\kappa)}{I_\nu(\kappa)}]\, K_\nu(\kappa_1)I_\nu(\kappa)}}$$

$$= \frac{1 - \dfrac{[\frac{\sigma}{2} I_\nu(\kappa_1) + \kappa_1 I_\nu'(\kappa_1)]\, K_\nu(\kappa)}{[\frac{\sigma}{2} K_\nu(\kappa_1) + \kappa_1 K_\nu'(\kappa_1)]\, I_\nu(\kappa)}}{1 - \dfrac{[\frac{\sigma}{2} I_\nu(\kappa_1) + \kappa_1 I_\nu'(\kappa_1)][\frac{\sigma}{2} K_\nu(\kappa) + \kappa K_\nu'(\kappa)]}{[\frac{\sigma}{2} K_\nu(\kappa_1) + \kappa_1 K_\nu'(\kappa_1)][\frac{\sigma}{2} I_\nu(\kappa) + \kappa I_\nu'(\kappa)]}}$$

Now

$$\frac{\sigma}{2} I_\nu(\kappa) + \kappa I_\nu'(\kappa) = (\frac{\sigma}{2} + \sqrt{m^2 + (\frac{\sigma}{2})^2}\ ) I_\nu(\kappa) + \kappa I_{\nu+1}(\kappa) \geq 0$$

$$\frac{\sigma}{2} K_\nu(\kappa) + \kappa K_\nu'(\kappa) = (\frac{\sigma}{2} - \sqrt{m^2 + (\frac{\sigma}{2})^2}\ ) K_\nu(\kappa) - \kappa K_{\nu+1}(\kappa) \leq 0$$

Furthermore

$$\frac{\sigma}{2} I_\nu(\kappa) + \kappa I_\nu'(\kappa) \geq \frac{\sigma}{2} I_\nu(\kappa_1) + \kappa_1 I_\nu'(\kappa_1)$$

for $\kappa_1 \leq \kappa$

$$\frac{\sigma}{2} K_\nu(\kappa) + \kappa K_\nu'(\kappa) \leq \frac{\sigma}{2} K_\nu(\kappa_1) + \kappa_1 K_\nu'(\kappa_1)$$

hence we obtain

$$\hat{E}_1 \geq 0$$

$$\hat{E}_2 = \frac{K_m(\kappa)\ [1 - \dfrac{I_m(\kappa)K_m'(\kappa_2)}{K_m(\kappa)I_m'(\kappa_2)}]}{\kappa K_m'(\kappa)\ [1 - \dfrac{I_m'(\kappa)K_m'(\kappa_2)}{K_m'(\kappa)I_m'(\kappa_2)}]}$$

Since

$$I_m(z) \geq 0 \qquad I_m'(z) \geq 0$$

$$z = \kappa,\ \kappa_2$$

$$K_m(z) \geq 0 \qquad K_m'(z) \leq 0$$

and

$$I_m'(\kappa_2) \geq I_m'(\kappa)$$

$$\text{for } \kappa \leq \kappa_2$$

$$K_m'(\kappa_2) \leq K_m'(\kappa)$$

therefore

$$E_2 \leq 0$$

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The author was born in Canton, China on November 8, 1944. In 1947, his family moved to Macao, where he received his education in elementary and high schools. In 1962, he went to Taipei, Taiwan to enroll as an undergraduate student in the Department of Civil Engineering at the National Taiwan University. Shortly after receiving his B.S. degree in June, 1966, he began serving as a full-time teaching and research assistant in the same department until August, 1969. With an intention of pursuing further studies, he came to the United States on an assistantship from the Department of Engineering Science and Mechanics at the University of Florida in September, 1969, and received his M.E. degree in Engineering Mechanics in December, 1970. In December, 1971, he married Hope Chen, who was a graduate student in the Botany Department at the same university. Their son, Hou-pung, was born on September 18, 1973. The author has now been working for his Ph.D. in the Department of Engineering Sciences at the University of Florida and expects to graduate in December, 1974.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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